Various Aspects of Quantum Measurements and Their Role in Quantum Information Processing Tasks

Thesis Submitted For The Degree Of

DOCTOR OF PHILOSOPHY (SCIENCE)

in

Physics (Theoretical)

by

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DEPARTMENT OF PHYSICS

JADAVPUR UNIVERSITY

2025

To my loving mother

Reba Das

&

My loving brother

Barun Das

Acknowledgement

I would like to express my deepest gratitude to all those who have supported and inspired me throughout my Ph.D. journey.

First and foremost, I would like to express my utmost gratitude to my supervisor, Prof. Archan S. Majumdar, for his unwavering support, constant encouragement and belief in my abilities. I must acknowledge the complete freedom he gave me regarding my choice of research topic and the way to deal with it.

It is my pleasure to thank my closest collaborators – Dr. Debashis Saha, Dr. Debarshi Das, Dr. Shiladitya Mal, Saheli Mukherjee, Dr. Bihalan Bhattacharya, and Prof. Dipankar Home for their support and intellectual exchange. Their stimulating discussions, insightful perspectives, and shared experiences have played a pivotal role in shaping my research and making this journey more enjoyable. I am fortunate to be part of a helpful and friendly group, consisting of Bivas Mallick, Saheli Mukherjee, Pritam Roy, Sudip Chakraborty, Subhankar Bera, Amit Kundu, and Shashank Shekhar Pandey. I must also thank Dr. Manik Banik and his group members (Sahil G. Naik, Ramkrishna Patra, Samrat Sen, Ananda G. Maity, Pratik Ghoshal, Ananya Chakraborty, and Kunika Agarwal) for the useful discussions from which I have learned a lot. My deepest appreciation goes to my family (my mother, Reba Das and brother Barun Das) and friends (Subhankar Bera, Pratik Ghoshal, Indrajit Ghose, Sayan Rout, Arnab Mukherjee, Arindam Biswas, Susanta Ghosh and Sidhartha Biswas) for their unwavering love and encouragement. I am grateful for their presence in my life and the motivation they have provided throughout this journey. I am also happy to get many jovial juniors-cum-friends like Rajdeep Biswas, Anirban Roychowdhury, Soumen Mandal, Ishita Jana, Shinjini Paul, Koushik Pradhan, Animesh Hazra, Soham Saha, Soham Sen, Ardhendu Paul, Sourav Sarkar and Prasanta Sarkar for making my campus life more enjoyable.

I am also thankful to Prof. Guruprasad Kar and Prof. Somshubhro Bandyopadhyay for piquing my interest in the field of Quantum information theory.

I am also grateful to have discussions with Dr. Som Kanjilal, Dr. Souradeep Sasmal, and Dr. Arkaprabha Ghoshal during my initial stage of learning the subject, and it makes my understanding better.

I am thankful to my teachers, Dr. Debapriyo Syam, Dr. Rajarshi Ray, Kalyan Majumdar, Samir Sarkar, Sougata Ghosh, Pradyut Biswas, Subir Patra, Subir Das, Dr. Subir Sarkar and Sharmistha Majumdar for their guidance and continuous support in pursuing my dream.

I am thankful to all the office/technical staff, the canteen staff, the hostel cooking team, the housekeeping staff, and gardeners of SNBNCBS for giving a pleasant experience during the PhD tenure.

I am thankful to my host institute, S. N. Bose National Centre for Basic Sciences (Kolkata), for supporting me financially and for providing me all the technical facilities throughout my Ph.D. period.

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To be submitted as per this format

CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled "**Various aspects of quantum measurements and their role in quantum information processing tasks**" Submitted by Sri / Smt. **Arun Kumar Das** who got his / her name registered on **23/06/2022** for the award of Ph. D. (Science) Degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Archan S. Majumdar** and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

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Abstract

This thesis aims to understand the various aspects of quantum measurements and investigate how these properties are crucial in information-processing tasks. To be more specific, we consider incompatible measurements, which are measurements that can not be performed jointly on a single system. It is purely a quantum effect with no classical analogue. We showed that measurement incompatibility is a necessary resource for achieving any advantage over classical strategies in prepare-and-measure communication tasks. We also proposed a practically implementable method to witness measurement incompatibility — a direction that has significant experimental relevance. We introduced a hierarchy of incompatibility classes, defined by classical operations applied to the inputs and outputs of measurement devices. This framework allows one to assess and compare measurement devices based on the strength of measurement incompatibility and facilitating practical choices in designing quantum protocols. We have explored how entanglement can be recycled in sequential measurement networks. We showed that reusing a single entangled state across multiple parties can be more resource-efficient than deploying independent copies — a finding with direct relevance to quantum networks and distributed quantum tasks.

List of publications and pre-prints

Publications and pre-prints forming part of the Thesis:

1. *"Resource-theoretic efficacy of the single copy of a two-qubit entangled state in a sequential network,"* **Arun Kumar Das**, Debarshi Das, Shiladitya Mal, Dipankar Home, A. S. Majumdar, Quantum Information Process 21, 381 (2022).

2. *"Measurement incompatibility and quantum advantage in communication,"* Debashis Saha, Debarshi Das, **Arun Kumar Das**, Bihalan Bhattacharya, A. S. Majumdar, Physical Review A 107, 062210 (2023).

3. *"An operational approach to classifying measurement incompatibility,"* **Arun Kumar Das**, Saheli Mukherjee, Debashis Saha, Debarshi Das, A. S. Majumdar, arXiv:2401.01236 [quant-ph] (2024).

Communicated articles not included in the Thesis:

1. *"Measurement-Device-Independent Certification of Schmidt Number,"* Saheli Mukherjee, Bivas Mallick, **Arun Kumar Das**, Amit Kundu, Pratik Ghosal, arXiv:2502.13296 [quant-ph] (2025).

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Chapter 1

Introduction

1.1 Prelude

One of the primary quests in science is to have a better understanding of nature. Scientists of the late 19th century had a firm belief that with the help of classical physics, i.e., classical mechanics, electrodynamics, thermodynamics and statistical mechanics, all the natural phenomena could be satisfactorily explained. However, this satisfaction of the scientists did not last for too long, and there appeared new natural phenomena and experimental findings, e.g., the black-body radiation [1], the Michelson-Morley experiment [2], the photoelectric effect [3], etc., that classical physics failed to give satisfactory explanations. To explain those phenomena, a new physics was born, which is "Quantum theory", and the founding fathers were Max Planck, Niels Bohr, Albert Einstein, Erwin Schrödinger, P.A.M. Dirac, Heisenberg, Max Born and many others. It is the most successful theory that revolutionised the world by making fundamental discoveries, e.g., the discovery of transistors [4], superconductivity [5], superfluidity [6, 7], invention of LASER [8] and many more. This has brought about the first quantum revolution [9]. Then the concept of quantum theory is applied in entirely different fields like information theory and computer science, and a new subject of research has emerged, which is called quantum information and quantum computation [10], and this introduces the second quantum revolution [9]. The objectives in this field of study are — (1) to ensure secrecy of communication which is governed by the laws of quantum physics [11–13], present-day classical communication is not perfectly secure as the security is based upon computationally hard problem, which our existing super computer takes an enormous time to break, however, a quantum computer can do so immediately and the security will be in danger; (2) to build quantum computer which can do faster and efficient computation compared to the classical computer; (3) to construct numerous information-processing tasks, e.g., quantum teleportation [14], quantum super dense coding [15], random access codes [16], etc., where quantum resources proved to be advantageous compared to the classical counterpart. All these exciting applications make the field fascinating, and many new opportunities remain open.

The primary objective of this thesis is to study the various aspects of quantum measurements and how they play a crucial role in quantum information processing tasks. In quantum theory, the concept of measurement is very different from the classical theory [17]. In classical theory, measurement just reveals a pre-existing value of the observable of the system, and the value can be obtained deterministically if all the factors are known — this is called reality, whereas quantum measurements are probabilistic, and the concept of reality is not respected here [18]. In quantum theory, there are incompatible measurements that can not be performed jointly [19, 20]. These incompatible measurements, which may seem to be an obstacle, are very useful for various quantum communication tasks [21–24]. The plan of the thesis is the following: in this chapter, I first give a general motivation behind the research in the direction of quantum information science. I then present the preliminary concepts like the Postulates of quantum theory in Sec. 1.2, wherein we discuss the fundamental concepts of quantum states, measurements, quantum evolution, and the concept of separable and entangled states. I then discuss the incompatibility of quantum measurements in Sec. 1.3 — that is widely used throughout the thesis. I discuss the concepts of entanglement witness in Sec. 1.4, Random access codes (RAC) in Sec. 1.5. The concepts of Bell non-locality and quantum steering, which are closely related to measurement incompatibility, are discussed in Sec. 1.6 and in Sec. 1.7, respectively. After discussing the pre-requisites, I present the main findings of the thesis.

In Chapter (2), I discuss the role of measurement incompatibility in quantum communication tasks, which is based on one of our research works [24]. Any communication task can be visualised as follows: this is a co-operative game between two players, Alice and Bob, who are in distant locations and get inputs x and y respectively from a referee; these inputs are some random variables. The players are ignorant about the other player's input. Alice wants to convey the information regarding her input to Bob, but she is constrained to send only a limited amount of communication. Bob, after receiving the message from Alice and the input from the referee, sends an output to the referee. The referee then gives a reward to the team (Alice and Bob) depending on the rules of the game. The final goal of the team is to maximise this reward. We have shown that any quantum strategy that outperforms the best classical strategy in this communication task, Bob must perform incompatible measurements in his system. Thus, measurement incompatibility is a necessary resource for obtaining quantum advantage in any communication task. We then give an explicit example of this kind of communication task, i.e., Random Access Codes (RAC) task and using this task, we give a method to witness incompatible measurements, which is very important from the practical point of view, as incompatible measurements are potential candidates for several quantum communication tasks.

In Chapter (3), I discuss the operational classifications of measurement incompatibility, which is based on our work [25]. In this work we give a hierarchy of different layers of measurement incompatibility based on the classical operations performed on the input and output of the measurement device. This hierarchy will facilitate the wise choice of incompatible measurements for practical purposes based on their strength of incompatibility. We also consider the unavoidable presence of noise in the real world and study how it weakens the degree of incompatibility.

Chapter (4) is based on our work [26], here we discuss how entanglement can be recycled in a sequential network scenario, such that it will be economical in terms of resource consumption compared to the non-sequential counterpart, where entanglement is not recycled; rather, multiple copies of the entangled states are used. This finding is especially relevant for quantum networks, where entanglement distribution is expensive and fragile.

Finally, in Chapter (5), I summarise the findings of this thesis and discuss the possible open problems that are worth pursuing in future.

1.2 Postulates of Quantum Theory

Quantum theory is a mathematical framework to describe the physical world. The main distinction of quantum theory from other theories is that it comes with some abstract ad hoc mathematical postulates which are not motivated by physical principles. The founding fathers of quantum theory came up with these postulates from numerous trial-and-error methods to explain the experimental observations that classical theories failed to explain.

Postulate 1 : State

Associated with every quantum mechanical system S, there is Hilbert space $\mathcal{H}_S$. A system is completely described by a positive operator σ with unit trace, known as 'density operator', acting on $\mathcal{H}_S$.

The most elementary quantum mechanical system is characterized by a Hilbert space of dimension two, and is referred to as a *qubit*. A qubit may correspond to the polarisation degree of freedom of a single photon or the spin degree of freedom of a spin-$\frac{1}{2}$ particle. The associated state space is given by the set $\mathcal{D}(\mathcal{H}_S)$, consisting of all positive, trace-one, self-adjoint linear operators acting on the Hilbert space $\mathcal{H}_S$. Formally, the elements $\sigma \in \mathcal{D}(\mathcal{H}_S)$ satisfy the following conditions:

1. $\sigma \geq 0$,
2. $\mathrm{Tr}(\sigma) = 1$,
3. $\sigma = \sigma^\dagger$.

Since we are working over a complex Hilbert space, positivity ($\sigma \geq 0$) implies Hermiticity ($\sigma = \sigma^\dagger$). The set $\mathcal{D}(\mathcal{H}_S)$ forms a convex and compact subset of the space of linear operators on $\mathcal{H}_S$. The elements of this set, known as *density operators*, can be further classified into two categories: *pure states* and *mixed states*.

Pure state: A quantum state $\sigma \in \mathcal{D}(\mathcal{H}_S)$ is called a pure state if it cannot be represented as a convex combination of other elements in $\mathcal{D}(\mathcal{H}_S)$. In other words, pure states correspond to the extreme points of the convex set of states $\mathcal{D}(\mathcal{H}_S)$. Rank-one projection operators of the form $|\psi\rangle\langle\psi| \in \mathcal{D}(\mathcal{H}_S)$, where $|\psi\rangle$ is a unit-norm vector in $\mathcal{H}_S$, represent pure states. Thus, pure states are commonly denoted by a unit vector $|\psi\rangle \in \mathcal{H}_S$. The necessary and sufficient condition for a state σ to be pure is $\sigma^2 = \sigma$, which also implies that $\mathrm{Tr}(\sigma^2) = 1$.

Mixed state: A density operator $\sigma \in \mathcal{D}(\mathcal{H}_S)$ that can be written as a convex combination of pure states is referred to as a mixed state. Such a state can be expressed as $\sigma = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where each p_i represents the probability associated with the pure state $|\psi_i\rangle$. Unlike pure states, mixed states satisfy $\mathrm{Tr}(\sigma^2) < 1$.

A key non-classical feature of quantum mechanics is that the decomposition of a mixed state into pure states is generally *not unique*. This contrasts with classical theories, where state spaces are simplices and each mixed state has a unique convex decomposition. In quantum theory, a single mixed state can admit infinitely many distinct convex decompositions into pure states. Specifically, if σ is a mixed state, there may exist different ensembles $\{|\psi_m\rangle\}$ and $\{|\phi_n\rangle\}$ such that

$$\sigma = \sum_m p_m |\psi_m\rangle \langle\psi_m| = \sum_n q_n |\phi_n\rangle \langle\phi_n|,$$

where $\{p_m\}$ and $\{q_n\}$ are sets of probabilities satisfying $p_m \geq 0$, $q_n \geq 0$, and $\sum_m p_m = \sum_n q_n = 1$. In other words, different ensembles of pure states can correspond to the same mixed state.

Postulate 2 : Measurement

Measurements in quantum theory are described by a collection of operators $\{M_k\}$ acting on the state space of the system, $\mathcal{D}(\mathcal{H}_S)$ satisfying the condition $\sum_k M_k^\dagger M_k = \mathbb{I}$. When a measurement M corresponding to this set $\{M_k\}$ is performed on a quantum system in state ρ, the probability of obtaining the k^{th} outcome is given by

$$p(k\,|\sigma, M) = \mathrm{Tr}(M_k^\dagger M_k \sigma),$$

and the post-measurement state, conditioned on the occurrence of the k^{th} outcome, is

$$\sigma_k = \frac{M_k \sigma M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \sigma)}.$$

Projective measurement: A commonly used special case of quantum measurements is known as a projective measurement. A set of measurement operators $\{M_i\}_{i=1}^n$ forms a projective measurement if each operator satisfies $M_i^\dagger = M_i$ and $M_i M_j = M_i\, \delta_{ij}$ for all i, j. That is, the measurement operators are orthogonal projectors.

Each rank-one projector M_i can be associated with a unit vector $|\psi_i\rangle$ such that $M_i = |\psi_i\rangle \langle\psi_i|$. Consequently, the maximum number of distinct outcomes n equals the dimension of the Hilbert space on which the measurement acts. From this point onward, projective measurement operators will be denoted by P_i instead of M_i.

If $\{P_i\}$ is a collection of orthogonal projectors representing a projective measurement, then they satisfy the completeness relation $\sum_i P_i = \mathbb{I}$. By assigning a set of real numbers $\{\lambda_i\}$ to these projectors, one can define a Hermitian operator (observable) as

$$\hat{A} = \sum_{i=1}^n \lambda_i P_i = \sum_{i=1}^n \lambda_i |\psi_i\rangle \langle\psi_i|.$$

Thus, in the case of projective measurements, quantum observables are identified with self-adjoint operators on the system's Hilbert space.

When a measurement corresponding to an observable $\hat{A}$ is performed on a system in state $|\psi\rangle$, the possible outcomes are the eigenvalues $\{\lambda_i\}$ of $\hat{A}$. The probability of obtaining the i^{th} outcome λ_i is given by

$$p(i) = |\langle\psi|\psi_i\rangle|^2|,$$

and the post-measurement state collapses to the corresponding eigenvector $|\psi_i\rangle$. Therefore, the expectation value of the observable $\hat{A}$ in the state $|\psi\rangle$ is

$$\langle\hat{A}\rangle_\psi = \sum_i p(i)\lambda_i = \sum_i |\langle\psi|\psi_i\rangle|^2\lambda_i = \langle\psi|\hat{A}|\psi\rangle\,.$$

POVM: A *positive operator-valued measure* (POVM) represents the most general type of quantum measurement that can be implemented on a system. Formally, a POVM is a collection of positive semi-definite operators $\{E_i\}$, i.e., $E_i \geq 0$ for all i, and satisfying the completeness relation $\sum_i E_i = \mathbb{I}$. The individual elements E_i, often referred to as *effects*, determine the probabilities of different measurement outcomes. Given a quantum system in state σ, the probability of obtaining outcome i when performing the POVM $\{E_i\}$ is given by $p(i) = \mathrm{Tr}[\sigma E_i]$.

To relate POVMs to the standard measurement framework, one can introduce a set of measurement operators $\{M_i\}$ such that $M_i = \sqrt{E_i}$ and hence $E_i = M_i^\dagger M_i$. It is important to note that this decomposition is not unique; multiple sets of $\{M_i\}$ can correspond to the same POVM [10].

Moreover, any POVM can be interpreted as a projective measurement on an extended Hilbert space, a result known from Naimark's dilation theorem [27]. A natural question arises here: in what way does the POVM formalism differ from the conventional measurement postulates of quantum mechanics?

The standard measurement postulate comprises two components: (i) a rule for computing the probabilities of measurement outcomes, and (ii) a prescription for determining the post-measurement state of the system. However, in many practical scenarios, only the outcome probabilities are of interest, while the post-measurement state can be disregarded. In such contexts, the POVM formalism offers a particularly convenient and powerful framework.

Postulate 3: Dynamics

The time evolution of a closed quantum system is governed by a unitary transformation. If the system evolves from state $\sigma(t_0)$ at time t_0 to state $\sigma(t_1)$ at a later time t_1 ($t_1 > t_0$), the evolution is described by a unitary operator $U(t_0,t_1)$ depending only on t_0 and t_1 such that

$$\sigma(t_1) = U(t_0,t_1)\,\sigma(t_0)\,U^\dagger(t_0,t_1)$$

This discrete-time description arises from the continuous-time dynamics dictated by the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

where H denotes the Hamiltonian of the closed system and $\hbar$ is the reduced Planck constant. The solution to this equation gives the state evolution from $|\psi_0\rangle$ at time t_0 to $|\psi_1\rangle$ at time t_1 as:

$$|\psi_1\rangle = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t_1} H\,dt\right]|\psi_0\rangle$$

While the evolution of closed systems is always unitary, open systems—i.e., systems interacting with an environment—undergo more general transformations described by *quantum operations* or *channels*. A quantum channel is a completely positive, trace-non-increasing, linear map from the set of density operators on one Hilbert space to another [10]. In the case where input and output Hilbert spaces coincide, a trace-preserving quantum channel $\mathcal{N} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ can be expressed in Kraus form as:

$$\mathcal{N}(\sigma) = \sum_k E_k\,\sigma\,E_k^\dagger \quad \forall\sigma \in \mathcal{D}(\mathcal{H}),$$

where the Kraus operators $\{E_k\}$ satisfy the completeness condition $\sum_k E_k^\dagger E_k = \mathbb{I}$. As with POVMs, the representation of a quantum channel in terms of Kraus operators is not unique [10].

Postulate 4: Composite Systems

The Hilbert space associated with a composite quantum system is given by the tensor product of the Hilbert spaces corresponding to its subsystems. That is,

$$\mathcal{H}_{1,2,\cdots,n} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n.$$

An n-partite quantum state is described by a density operator $\rho_{1,2,\cdots,n}$ acting on the composite Hilbert space $\mathcal{H}_{1,2,\cdots,n} = \bigotimes_{i=1}^{n} \mathcal{H}_i$. If the global state can be expressed as a tensor product of individual states, i.e.,

$$\sigma_{1,2,\cdots,n} = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n, \quad \text{with } \sigma_i \in \mathcal{D}(\mathcal{H}_i),$$

then it is referred to as a *product state*. A state that is a convex combination of such product states is called a *separable state*.

To illustrate this more concretely, consider the case of a bipartite system consisting of two subsystems A and B. A bipartite separable state σ_{AB}^{sep} has the form:

$$\sigma_{AB}^{\text{sep}} = \sum_j p_j \sigma_A^j \otimes \sigma_B^j, \tag{1.1}$$

where each $p_j \geq 0$ and $\sum_j p_j = 1$, and $\sigma_A^j \in \mathcal{D}(\mathcal{H}_A)$, $\sigma_B^j \in \mathcal{D}(\mathcal{H}_B)$.

The set of all separable states forms a strict subset of the total state space $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Denoting the set of separable states as $\text{sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have:

$$\text{sep}(\mathcal{H}_A \otimes \mathcal{H}_B) \subset \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

Any state $\sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ that does *not* belong to the set of separable states is called an *entangled state*. In other words, an entangled state cannot be expressed as a convex combination of product states as shown in Eq. (1.1).

1.3 Measurement Incompatibility

A set of measurements is called incompatible or non-jointly measurable if their outcome probability statistics can not be recovered from a single parent measurement outcome statistics [28, 20]. Let us consider a set of measurements $\{M_{z|x}\}_{z,x}$, where x denotes different measurements, and z denotes the corresponding outcomes. The measurement set is said to be compatible if there exists a parent POVM, G_λ, and classical post-processing $\{p(z|x,\lambda)\}$ for each x such that

$$\forall z, x, \quad M_{z|x} = \sum_{\lambda} p(z|x,\lambda) G_{\lambda}, \tag{1.2}$$

where $0 \leq p(z|x,\lambda) \leq 1$, and $\sum_z p(z|x,\lambda) = 1$, for all x, λ [29]. As a special case, if the operators are projectors, then the two measurements are jointly measurable when their corresponding operators commute.

Now without loss of generality the post-processing $\{p(z|x,\lambda)\}$ can be replaced by deterministic post-processing $\{k(z|x,\lambda)\}$, i.e., $k(z|x,\lambda) \in \{0,1\}$ [30]. Now the problem of verifying whether a set of measurements is incompatible can be cast as a semi-definite programming as follows:

$$\begin{gathered} \max_{\{G_\lambda\},\{k(z|x,\lambda)\}} \xi \\ \text{subject to: } 0 \leq G_\lambda \leq \xi \mathbb{1}, \\ M_{z|x} = \textstyle\sum_\lambda k(z|x,\lambda) G_\lambda, \sum_\lambda G_\lambda = \mathbb{1}. \end{gathered} \tag{1.3}$$

If the maximum value of ξ becomes negative, then the corresponding measurements are incompatible.

1.4 Witness of entanglement

Whether a state is entangled or not is difficult to answer; it is an NP hard problem [31–33]. If the partial transposition of a density matrix (σ^{T_A} or σ^{T_B}) has at least one negative eigenvalue, then the state is entangled for sure. The necessary criterion for a bi-partite state σ to be separable is $\sigma^{T_A} \geq 0$, also known as the PPT criterion, which A. Peres introduced in Ref.[34]. Subsequently, the Horodecki family proved that this condition is necessary as well as sufficient for systems with the dimension of the subsystems of A and B being ($d_A = 2$, $d_B = 2$), ($d_A = 2$, $d_B = 3$) and ($d_A = 3$, $d_B = 2$) [35]. There are entangled states which satisfy the PPT criterion are called bound entangled states. The negation of the PPT condition, which is also known as the NPT condition, can theoretically detect entanglement, but it is not practically realisable.

In the bipartite state space, the set of separable states forms a convex compact subset of the set of all density states, i.e., sep$(\mathcal{H}_A \otimes \mathcal{H}_B) \subset \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Thus, due to the Hann-Banach separation theorem [36], for every entangled state, a hyperplane always exists that differentiates it from the set of separable states, see Fig. 1.1. And, this hyperplane corresponds entanglement witness operator ($\mathcal{W}$) for that entangled state σ_e, satisfying the following relations [37, 38]:

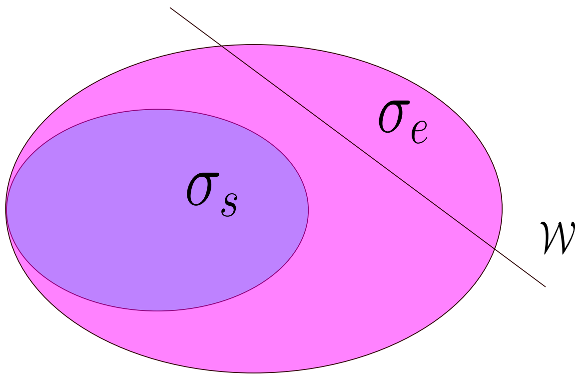

FIG. 1.1 Set inclusion relation of quantum states, separable states and entangled states. Also, the concept of entanglement witness operator ($\mathcal{W}$) from the Hann-Banach theorem.

1. $Tr(\sigma_s \mathcal{W}) \geq 0,$ for all separble states $\sigma_s \in \text{sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$.
2. $Tr(\sigma_e \mathcal{W}) < 0$ for entangled state σ_e.
3. $\mathcal{W} = \mathcal{W}^\dagger$.

In most of the practical purposes, we expect a particular state σ to be given, which may sometimes contain some noise; our job is to verify whether σ is entangled or not. From the experimental point of view, it is important to implement the witness operator in terms of local observables of Alice and Bob so that by performing measurements on their respective labs and recording the statistics and exchanging classical communications, they can compute the expectation value of the witness operator. For example, if $\mathcal{W} = \sum_{i=1}^{m} c_i A_i \otimes B_i$, the local measurements of Alice are $\{A_i\}$ and for Bob are $\{B_i\}$. The expectation value of $\mathcal{W}$ is the weighted sum of the products of expectation values of the local measurements, i.e., $\langle \mathcal{W} \rangle = \sum_{i=1}^{m} c_i \langle A_i \rangle \langle B_i \rangle$. One natural objective is to minimise m, i.e., the number of measurements Alice (Bob) has to perform to compute the expectation value of the witness operator.

An entanglement witness operator exists for every entangled state; however, finding an optimal entanglement witness operator is not easy [39]. Consider the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The best entanglement witness operator (optimal witness) for this state is given by [39],

$$\mathcal{W} = \frac{1}{4}\Big(\mathbb{I}\otimes\mathbb{I} - \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\Big). \tag{1.4}$$

In the realistic scenario, during the preparation of $|\psi^+\rangle$, some impurity (noise) creeps in, and the resultant state may look like

$$\sigma = p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2}, \tag{1.5}$$

where $p \in (0,1]$; $\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2}$ corresponds white noise. For the state σ also, the entanglement witness operator $\mathcal{W}$ serves as the optimal witness operator [39].

1.5 Random Access Codes (RAC)

Random Access Code (RAC) is a particular communication task [40, 41]. It is a cooperative game played between Alice (sender) and Bob (receiver). Lets consider Alice has a collection of n-dit string, $\{\vec{x}\} \equiv \{(x_1, x_2, \cdots, x_n)\}$ where x_i can take d_i possible values i.e., $x_i \in \{0,1,\cdots,d_i-1\}$, with $i \in \{1,2,\cdots,n\}$. Alice gets a random input $x \in \{\vec{x}\}$ and Bob gets a random input $y \in \{1,2,\cdots,n\}$. Neither player has any information about the other player's input. Alice can help Bob, but is restricted to sending only a d-level classical or quantum system. Bob has to guess the value of x_y, and he gives an output z. The probability of correctly guessing the y^{th} element of the n-dit string is given by: $p(z = x_y|\vec{x}, y)$. Now, two figures of merit are used to assess the success metric of the RAC. One is the "worst case success probability", defined as:

$$W(n, \vec{d}, d) = \min_{\vec{x},y}\{p(b_y = x_y|\vec{x}, y)\}, \tag{1.6}$$

and another is the "average success probability", defined as:

$$S(n, \vec{d}, d) = \frac{1}{nd^n}\sum_{\vec{x},y} p(z = x_y|\vec{x}, y). \tag{1.7}$$

The goal of the RAC is to maximise these success probabilities. In our work, we mostly consider the "average success probability".

There are some instances where it is observed that the average success probability can be obtained using a suitable quantum strategy (suitable quantum systems communicated by Alice and suitable quantum measurements performed by Bob) is higher than the average success probability obtained using the best classical strategy (classical system communicated by Alice and classical measurement performed by Bob) [42, 24]. This suggests that the quantum communication protocol is advantageous in the RAC task over classical communication. We discuss in the Chapter (2) the reason behind this quantum advantage over the classical counterpart in this RAC task, or more generally, in any communication task.

1.6 Non-locality and Bell's theorem

In classical physics, if two objects are far apart and there is no interaction between them, the choices of measurements performed on one object will not affect the outcomes of the measurements performed on the other object. This is called the principle of “locality”. Another important concept in classical physics is the concept of “reality”. It suggests that it is possible in principle to know the values of observable quantities of a physical system without disturbing the system. The two concepts “locality” and “reality” of classical physics are commonly known as “local realism”. We can now ask the following question: is the principle of local realism always obeyed in quantum physics? The answer is “No”. If we take two entangled quantum objects, the measurement choice on one object can affect the outcomes of the measurements on the other, irrespective of their physical separation. Also, there are incompatible observables in quantum theory (e.g., the position and momentum of an electron) whose values can not be known simultaneously. This problem bothers one of the founding fathers of quantum theory, Albert Einstein, Boris Podolsky, and Nathan Rosen [18]. Initially, whether quantum theory was a local realistic theory was a matter of philosophical debate. John Bell had first shifted this philosophical debate to an experimentally testable hypothesis [43]. In his paper [43], he shows that every theory consistent with the principle of local realism must satisfy a precise statistical inequality known as the Bell inequality. Quantum theory violates this Bell inequality and is thus not a local realistic theory, also sometimes loosely called the non-local theory. It was later experimentally confirmed by Alain Aspect [44].

Bell inequality: Consider two spatially separated parties, Alice and Bob, each with a black box that receives input and produces output. Let the input for Alice be denoted as x, and that for Bob is y and the output for Alice is a and for Bob is b. In the simplest non-trivial case $x,y \in \{0,1\}$ and $a,b \in \{+1,-1\}$. When Alice gets input $x = 0$, measurement A_0 is performed, and for $x = 1$, measurement A_1 is performed on her system, and the corresponding measurements for Bob are denoted by B_0 and B_1. Bell's theorem says that any theory that obeys the local realism principle must satisfy the following inequality called CHSH inequality [1]:

$$|\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle| \leq 2, \tag{1.8}$$

in the above-mentioned scenario. Where, $\langle A_i B_j \rangle = \sum_{a,b} ab P(ab|A_i B_j)$ denotes the expectation value corresponding to the measurements A_i and B_j. The local realism condition simply means the existence of some pre-shared strategy denoted by the variable λ, which causally influences the outcomes of Alice and Bob. Mathematically,

$$P(ab|A_i, B_j) = \int \mu(\lambda) P(a|A_i, \lambda) P(b|B_j, \lambda) d\lambda, \tag{1.9}$$

where, $\mu(\lambda)$ is the probability distribution over the local variable λ. Now the important point is that there exist entangled states and suitable measurement choices for Alice and Bob such that the joint statistics violate the Bell inequality (1.8) [43, 46, 47]. Thus, quantum theory is inconsistent with the local realism principle. For example, consider the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$ shared between Alice and Bob and take $A_0 = \sigma_z$, $A_1 = \sigma_x$, $B_0 = \frac{\sigma_z + \sigma_x}{\sqrt{2}}$ and $B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$. Using the Born rule, the value of the CHSH expression becomes $2\sqrt{2}$. This is notably also the maximum possible violation of CHSH inequality attainable in quantum theory, known as 'Cirel'son's bound' [48].

1.7 Quantum steering

Quantum steering is a non-local correlation which is stronger than quantum entanglement and weaker than Bell non-locality [49, 50]. If two spatially separated parties, Alice and Bob share a bipartite state, steering is the task, where Alice at her wish, can change the state ensemble on the Bob's part by suitably performing measurements in her side. A bi-partite state is steerable if the en-

[1] CHSH stands for John Clauser, Michael Horne, Abner Shimony, and Richard Holt, who gave this inequality in their paper [45], it is a particular type of Bell inequality.

semble prepared at Bob's side does not a have a Local Hidden State (LHS) model. Let's consider the state ρ_{AB} is shared and the measurements on Alice side is represented by the POVM: $\{M_{a|x}\}$, if Alice gets an outcome: a corresponding to the measurement setting: x, then the unnormalised state on Bob's side is given by $\rho_{a|x} = \mathrm{Tr}[(M_{a|x} \otimes I)\rho_{AB}]$. The state is unsteerable if

$$\rho_{a|x} = \int d\lambda\, \mu(\lambda)\, p(a|x,\lambda)\, \rho_\lambda \tag{1.10}$$

for all a, x. In this case Alice is not able to steer Bob by performing suitable measurements on her side. The Eq. (1.10) is known as the LHS model, which can be understood in this way: A source is producing a classical random variable λ and a quantum random state ρ_λ with probability $\mu(\lambda)$ which are sent to Alice and Bob respectively, Alice can then relabel λ by a after receiving an input x with the probability $p(a|x,\lambda)$, and Eq. (1.10) is the updated state of Bob on knowing the outcome a and the measurement setting x of Alice.

Measurement incompatibility has a one-to-one correspondence with steering [21, 22, 51]. Compatible measurements will never show steering, and for every set of incompatible measurements, it is possible to get a suitable entangled state (maximally entangled state) that exhibits steering.

Chapter 2

Measurement incompatibility and quantum advantage in communication

2.1 Introduction

In quantum theory, a set of quantum measurements is called incompatible if these measurements cannot be performed simultaneously on a single copy of a quantum system [20]. The best known example of quantum incompatibility pertains to the measurements of position and momentum of a quantum mechanical particle that cannot be measured simultaneously with arbitrary precision. The notion of measurement incompatibility is an inherent property of quantum theory that differentiates it from classical physics. Quantum measurement incompatibility is at the root of demonstrating various fundamental quantum aspects such as, Bell-nonlocality [52, 53], Einstein–Podolsky–Rosen steering [54–58], measurement uncertainty relations [59–61], quantum contextuality [62, 63], quantum violation of macrorealism [64, 65], and temporal and channel steering [66–68].

Bell-violation is the most compelling operational witness of incompatible measurements since it relies only on the input-output statistics of bipartite systems [53, 69, 70]. Measurement incompatibility can also be witnessed through Einstein–Podolsky–Rosen steering [54, 55, 57, 71, 72]. These protocols, however, rely on entanglement. Recently, witnessing of quantum measurement incompatibility in the prepare-and-measure scenario based on a state discrimination task [73] has been proposed. It is particularly noteworthy that measurement incompatibility is necessary but not sufficient for Bell-violations employing fully untrusted devices [74, 75], whereas incompatibility is shown to be necessary as well as sufficient in steering with one-sided trusted devices [55, 57], and in state discrimination tasks with fully trusted preparations [76] (see also [77–79]).

Nonetheless, the generic link between measurement incompatibility and nonclassical correlations in the simplest prepare-and-measure scenario is still not fully explored. This work is motivated towards filling this crucial gap in the literature. Moreover, the results presented here address the issue as to whether incompatible quantum measurements are necessary for probing quantum advantage in any communication scenario. Apart from answering this fundamental question in the affirmative, this work further aims to provide an operational witness of incompatibility for any set of quantum measurements of an arbitrary setting - any set of arbitrary numbers of measurements acting on an arbitrary but finite dimension wherein different measurements have different arbitrary numbers of outcomes.

We arrange the rest of this chapter as follows. In Sec. 2.2, we show that incompatible measurements by the receiver are necessary for quantum advantage in any communication task. Next, considering RACs, we present an operational witness of measurement incompatibility of an arbitrary set of quantum measurements in Sec. 2.3. Relationships between different types of probability distributions in communication tasks are depicted in Sec. 2.4. Finally, we conclude with a short discussion in Sec. 2.5.

2.2 Incompatibility is necessary for quantum advantage in communication tasks

We will now show that incompatibility is necessary for quantum advantage in any communication task in the prepare-and-measure scenario. The prepare-and-measure scenario is the fundamental building block of any experimental scenario, where a preparation device on obtaining an input x from a random variable $\mathcal{X}$, i.e., $x \in \mathcal{X}$, prepares a physical (classical or quantum) system, and the physical system is sent to a measuring device that performs a particular measurement on the system depending upon the input $y \in \mathcal{Y}$, and gives an outcome b_y. When nothing is known about the internal functioning of the preparing and measuring device, the experimental statistics can be best explained by the probabilities, $\{p(b_y|x,y)\}$.

Communication task in the prepare-and-measure scenario: The most general communication task in the prepare-and-measure scenario consists of three parties: two collaborative players, Alice and Bob, and a neutral referee. Alice and Bob get inputs $x \in [l]$ and $y \in [n]$ respectively from the referee (here we use the notation: $[k] \equiv \{0,1,2,\cdots,k-1\}$). Initially, they don't know exactly the other players' input but they have an idea from which set it is coming. Before the game starts they can agree upon some pre-shared strategies. But once the game starts they can only have a limited amount of communication. Alice is allowed to help Bob regarding the information of her input by sending a $d-$ level physical system (classical or quantum) to Bob. Bob then gives an output b_y on receiving the inputs from the referee and the physical system from Alice. The referee gives the team a payoff, and the figure of merit of this communication task depends solely on the probability distributions, $\{p(b_y|x,y)\}$.

In classical communication, they can use unlimited pre-shared randomness λ with some probability distribution $\pi(\lambda)$, and, therefore, any typical probability can be expressed as

$$p(b_y|x,y) = \sum_{m=1}^{d} \int_{\lambda} \pi(\lambda) p_a(m|x,\lambda) p_b(b_y|y,m,\lambda)\, d\lambda. \tag{2.1}$$

Here, $\{p_a(m|x,\lambda)\},\{p_b(b_y|y,m,\lambda)\}$ are encoding and decoding functions by Alice and Bob, satisfying non-negativity and

$$\sum_m p_a(m|x,\lambda) = \sum_{b_y} p_b(b_y|y,m,\lambda) = 1. \tag{2.2}$$

While in quantum communication, one can consider the set of probabilities with or without the pre-shared classical randomness. Suppose $\mathcal{B}(\mathbb{C}^d)$ stands for the space of all operators acting on d dimensional complex Hilbert space. In the former scenario with pre-shared randomness

$$\begin{aligned} p(b_y|x,y) = \int_{\lambda} \pi(\lambda) \mathrm{Tr}\left(\rho_{x,\lambda} M_{b_y|y,\lambda}\right) d\lambda, \\ \rho_{x,\lambda},\ M_{b_y|y,\lambda} \in \mathcal{B}(\mathbb{C}^d), \end{aligned} \tag{2.3}$$

where $\{\rho_{x,\lambda}\}$ is the quantum state sent by Alice upon input x and random variable λ, and $\{M_{b_y|y,\lambda}\}$ is the measurement executed by Bob for his input y and random variable λ. Without the pre-shared randomness the expression of the probabilities reduces to,

$$p(b_y|x,y) = \mathrm{Tr}(\rho_x M_{b_y|y}), \quad \rho_x, M_{b_y|y} \in \mathcal{B}(\mathbb{C}^d). \tag{2.4}$$

Measurement incompatibility and quantum advantage in communication

A communication scenario is specified by a set of natural numbers l, n, and $\vec{d} = (d_1, \cdots, d_n)$ such that $x \in [l]$, $y \in [n]$, $b_y \in [d_y]$. Given a scenario, we define the set of all probabilities obtainable by *d-dimensional classical communication,*

$$\mathcal{C}_d := \{p(b_y|x,y)\}, \tag{2.5}$$

where $p(b_y|x,y)$ is given by Eq. (2.1); the set of all probabilities in *d-dimensional quantum communication,*

$$\mathcal{Q}_d := \{p(b_y|x,y)\} \tag{2.6}$$

where $p(b_y|x,y)$ is given by Eq. (2.3); and the set of all probabilities in *d-dimensional quantum communication without randomness,*

$$\overline{\mathcal{Q}}_d := \{p(b_y|x,y)\} \tag{2.7}$$

where $p(b_y|x,y)$ is given by Eq. (2.4). In this work, we are interested in another two sets of probabilities. First, the set of all probabilities in *d-dimensional quantum communication restricted to compatible measurements,*

$$\mathcal{Q}_d^C := \{p(b_y|x,y)\}, \tag{2.8}$$

where $p(b_y|x,y)$ is given by Eq. (2.3) such that the set of measurements acting on d-dimensional quantum states used by Bob $\{M_{b_y|y,\lambda}\}$ is compatible, that is, there exists parent POVM $\{G_\kappa\}$ such that

$$\forall b_y, y, \lambda, \quad M_{b_y|y,\lambda} = \sum_\kappa P_{y,\lambda}(b_y|\kappa)\; G_\kappa \tag{2.9}$$

and

$$P_{y,\lambda}(b_y|\kappa) \geqslant 0 \; \forall y, \lambda, b_y, \kappa; \; \sum_{b_y} P_{y,\lambda}(b_y|\kappa) = 1 \; \forall y, \lambda, \kappa. \tag{2.10}$$

And second, the set of all probabilities in *d-dimensional quantum communication restricted to compatible measurements without shared randomness,*

$$\overline{\mathcal{Q}}_d^C := \{p(b_y|x,y)\}, \tag{2.11}$$

where $p(b_y|x,y)$ is given by Eq. (2.4), such that the set of measurements $\{M_{b_y|y}\}$ is compatible according to (1.2).

Result 1. *Given any scenario specified by* $(l,n,\vec{d})$,

$$\overline{\mathcal{Q}}_d^C \subseteq \mathcal{Q}_d^C = \mathcal{C}_d. \tag{2.12}$$

Thus, measurement incompatibility is necessary for any quantum advantage (with or without shared randomness) over classical communication.

Proof. The first relation, $\overline{\mathcal{Q}}_d^C \subseteq \mathcal{Q}_d^C$, follows from the definition of these two sets. The nontrivial part of this result is proving the equality.

Consider the case where Bob performs POVM measurements $\{G_\kappa\}$, which is the parent POVM of the measurement set $\{M_{b_y|y,\lambda}\}$. The Frenkel-Weiner theorem [80] implies that the set of input-output probabilities $p(\kappa|x)$ with a single quantum measurement on d-dimensional quantum states can always be reproduced by a suitable classical d-dimensional communication in the presence of shared randomness. In other words, $\forall \rho_{x,\lambda}$ there exists classical strategy $\tilde{\pi}(\tilde{\lambda}), p_a(m|x,\lambda,\tilde{\lambda}), p_b(\kappa|m,\tilde{\lambda})$ such that

$$\mathrm{Tr}(\rho_{x,\lambda} G_\kappa) = \sum_{m=1}^{d} \int_{\tilde{\lambda}} \tilde{\pi}(\tilde{\lambda}) p_a(m|x,\lambda,\tilde{\lambda}) p_b(\kappa|m,\tilde{\lambda})\ d\tilde{\lambda}. \tag{2.13}$$

Here, note that, in the scenario considered by Frenkel-Weiner [80], Bob does not receive any input y therein. That is why Bob's output κ depends only on the message m sent by Alice and classical shared randomness $\tilde{\lambda}$.

Let us now focus on the set of probabilities $\mathcal{C}_d$ wherein $(\lambda,\tilde{\lambda})$ is the pre-shared randomness. Take into account the following decoding function,

$$p_b(b_y|y,m,\lambda,\tilde{\lambda}) = \sum_\kappa P_{y,\lambda}(b_y|\kappa) p_b(\kappa|m,\tilde{\lambda}), \tag{2.14}$$

where $\{P_{y,\lambda}(b_y|\kappa)\}$ is the post-processing defined in (2.9) and $p_b(\kappa|m,\tilde{\lambda})$ is given in (2.13). One can check that this is indeed a valid decoding function.

Next, we show that an arbitrary $p(b_y|x,y) \in \mathcal{Q}_d^C$ can always be reproduced by a suitable classical strategy (2.1) involving pre-shared randomness $(\lambda,\tilde{\lambda})$ and the decoding function (2.14). With the help of (2.9), (2.13), and (2.14) in a subsequent order, we find

$$\begin{aligned}
& \int_\lambda \pi(\lambda)\mathbf{Tr}\left(\rho_{x,\lambda} M_{b_y|y,\lambda}\right)\, d\lambda \\
= \;& \sum_\kappa \int_\lambda P_{y,\lambda}(b_y|\kappa)\pi(\lambda)\mathbf{Tr}(\rho_{x,\lambda} G_\kappa)\, d\lambda \\
= \;& \sum_m \int_\lambda \int_{\tilde{\lambda}} \pi(\lambda)\tilde{\pi}(\tilde{\lambda}) p_a(m|x,\lambda,\tilde{\lambda}) \\
& \qquad\qquad \times \left(\sum_\kappa P_{y,\lambda}(b_y|\kappa) p_b(\kappa|m,\tilde{\lambda})\right) d\lambda\, d\tilde{\lambda} \\
= \;& \sum_m \int_\lambda \int_{\tilde{\lambda}} \pi(\lambda)\tilde{\pi}(\tilde{\lambda}) p_a(m|x,\lambda,\tilde{\lambda}) p_b(b_y|y,m,\lambda,\tilde{\lambda}) d\lambda d\tilde{\lambda}.
\end{aligned} \tag{2.15}$$

Therefore, an arbitrary probability distribution $p(b_y|x,y)$ obtainable from a compatible set of measurements can be reproduced by a suitable classical strategy, implying that $\mathcal{Q}_d^C \subseteq \mathcal{C}_d$. On the other hand, any classical strategy is always realized by quantum strategy with compatible measurements, i.e., $\mathcal{C}_d \subseteq \mathcal{Q}_d^C$. Therefore, these two sets are identical.

Finally, we remark that the figure of merit of any communication task is some arbitrary function of the probabilities $p(b_y|x,y)$, we can infer that any quantum advantage (with or without shared randomness) in such tasks over classical communication can be attained only if the set of measurements is incompatible. This completes the proof. □

One profound implication of *Result* 1 is that any communication task can serve as a witness of measurement incompatibility. However, in general, *measurement incompatibility is not sufficient for quantum advantage without preshared randomness* [81] – there exist incompatible qubit measurements such that the set of probabilities given by (2.4) for arbitrary quantum states is within $\mathcal{C}_2$ (see Sec. IV-A of [81]).

Another useful implication of *Result* 1 appears when we are interested in linear functions of $\{p(b_y|x,y)\}$. Communication tasks for which the figures of merit are linear functions of $\{p(b_y|x,y)\}$ are widespread due to their practical importance in quantum communication complexity tasks [82–84], quantum key distribution [85], quantum randomness generation [86, 87], quantum random access codes [88, 42], oblivious transfer [89, 84], and many other applications.

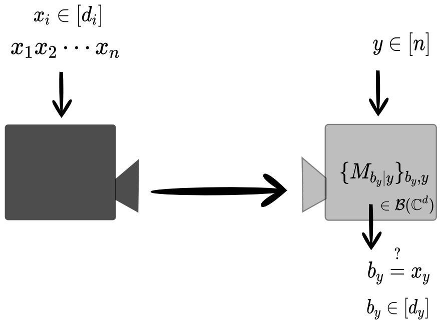

FIG. 2.1 An unknown measurement set of arbitrary settings, $\{M_{b_y|y}\}_{b_y,y}$, is provided; we only know the dimension (d) on which this set of measurements act. Our task is to certify the incompatibility of this set of measurements. Here, the notation $[k] := \{1, \cdots, k\}$ for any natural number k.

Fact 1. *Given any scenario specified by $(l, n, \vec{d})$, the maximum values of any linear function of $\{p(b_y|x,y)\}$ obtained within the three different sets $\mathcal{C}_d$, $\mathcal{Q}_d^C$, and $\overline{\mathcal{Q}}_d^C$ are the same.*

Proof. To find the optimum value of any linear function of $\{p(b_y|x,y)\}$, it is sufficient to consider classical strategy without shared randomness (see *Lemma*1 Eq. (B.1) in Appendix B for a detailed explanation). Now, all probability distributions $\{p(b_y|x,y)\}$, which are obtained from classical strategy without shared randomness, can always be realized within $\overline{\mathcal{Q}}_d^C$. Upon receiving the input x, Alice sends the quantum state ρ_x such that ρ_x is diagonal in the computational basis. Bob, upon receiving the input y and the diagonal state ρ_x, performs a fixed measurement $\{G_\kappa\}$ in the computational basis followed by some post-processing depending on y. This observation together with the relation $\overline{\mathcal{Q}}_d^C \subseteq \mathcal{Q}_d^C = \mathcal{C}_d$ implies the above fact. □

The generic relation among the sets $\overline{\mathcal{Q}}_d^C, \mathcal{Q}_d^C, \overline{\mathcal{Q}}_d, \mathcal{Q}_d, \mathcal{C}_d$ is further analyzed in Section 4.6. Next, we will propose incompatibility witness for an arbitrary set of measurements using a family of communication tasks, namely, the general version of RAC [16].

2.3 Incompatibility witness for any set of measurements of arbitrary setting

Take the most general form of a set of measurements. There are n measurements, defined by $\{M_{b_y|y}\}$ where $y \in [n]$ each of which has different outcomes, say, measurement y has d_y outcomes, that is, $b_y \in [d_y]$. These measurements are acting on d-dimensional quantum states where d is finite (see FIG. 4.1). In order to witness incompatibility of this set, we introduce the most general form of random access codes with Bob having this set of measurements. Alice gets a string of n dits $x = x_1x_2\cdots x_n$ randomly from the set of all possible strings in which $x_y \in [d_y]$ for all $y \in [n]$. While Alice communicates a d-dimensional classical or quantum system to encode the information about the obtained string, the task for Bob is to guess the y-th dit when y is chosen randomly. The figure of merit is the average success probability defined by the following linear function

$$S(n,\vec{d},d) = \frac{1}{n\prod_y d_y}\sum_{x,y} p(b_y = x_y|x,y) \tag{2.16}$$

which is fully specified by n, $\vec{d} = (d_1,d_2,\cdots,d_n)$, and d. Since Eq.(2.16) is a linear function of $p(b_y|x,y)$, by Fact 1, the maximum values over $\mathcal{C}_d$, $\mathcal{Q}_d^C$ and $\overline{\mathcal{Q}}_d^C$ are the same and denoted by $S_c(n,\vec{d},d)$. Precisely,

$$\begin{aligned} S_c(n,\vec{d},d) &= \max_{\{p(b_y|x,y)\}\in\mathcal{C}_d} S(n,\vec{d},d) \\ &= \max_{\{p(b_y|x,y)\}\in\mathcal{Q}_d^C} S(n,\vec{d},d). \end{aligned} \tag{2.17}$$

Hence, $S_c(n,\vec{d},d)$ can be evaluated by maximizing the average success probability either over all classical strategies, or over all quantum strategies involving compatible measurements only.

Whenever a set of measurements in the scenario specified by n, $\vec{d}$, d gives $S(n,\vec{d},d) > S_c(n,\vec{d},d)$ in the above-introduced general version of RACs, we can conclude that the measurements are incompatible. Hence, in order to witness measurement incompatibility, we need to know $S_c(n,\vec{d},d)$. Now we present an upper bound on $S_c(n,\vec{d},d)$ for arbitrary $n,\vec{d},d$.

Result 2. *The following relation holds true for arbitrary* $(n,\vec{d},d)$:

$$S_c(n,\vec{d},d) \leqslant \frac{1}{n} \times \min\left\{1 + \sum_{\substack{i,j\\ i<j}} \frac{d}{d_i d_j}\ ,\ n-1+\frac{d}{\prod_y d_y}\right\}. \tag{2.18}$$

This upper bound in Eq. (2.18) is obtained for $\mathcal{Q}_d^C$, that is, by taking the existence of a parent POVM of the measurements $\{M_{b_y|y}\}_{b_y,y}$ performed by Bob. The proof of this result is presented in Appendix A. When the outcome of all the measurements is the same, which is $d_y = \overline{d}$ for all y, the above bound simplifies to

$$S_c(n,\overline{d},d) \leqslant \frac{1}{n} \times \min\left\{1 + \frac{n(n-1)d}{2\overline{d}^2}\ ,\ n-1+\frac{d}{\overline{d}^n}\right\}. \tag{2.19}$$

Hence, in different types of RACs involving an arbitrary set of quantum measurements by Bob, if the average success probability exceeds the aforementioned upper bounds on S_c, then we can conclude that the measurements by Bob are incompatible.

On the other hand, whenever

$$d \leqslant \min_y d_y \tag{2.20}$$

we find out the exact value of $S_c(n,\vec{d},d)$. Say, k_i is the number of sets among $[d_1],\cdots,[d_n]$ such that dit $i \in [d_y]$. For example, consider the RACs with $n=4$ and $d_1=2$, $d_2=3$, $d_3=4$ and $d_4=3$. That is, Alice gets a string of four dits $x = x_1x_2x_3x_4$ randomly, where $x_1 \in [2]$, $x_2 \in [3]$, $x_3 \in [4]$ and $x_4 \in [3]$. In this case, $k_1=4$, $k_2=4$, $k_3=3$, and $k_4=1$. Also, we denote $d_{\max} = \max_y d_y$.

Result 3. *If* (2.20) *holds, then*

$$S_c(n,\vec{d},d) = \frac{1}{n\prod_y d_y}\sum\left[\left(\prod_{j=1}^{d_{\max}} C_{n_j}^{\alpha_j}\right)\max_{i=1,\cdots,d}\{n_i\}\right] \tag{2.21}$$

with

$$\alpha_j = k_j - \sum_{i=j+1}^{d_{\max}} n_i, \quad C_{n_j}^{\alpha_j} = \frac{\alpha_j(\alpha_j-1)\cdots(\alpha_j-n_j+1)}{n_j(n_j-1)\cdots 1}$$

and where the summation is taken over all possible integer solutions of the following equation

$$\sum_{i=1}^{d_{\max}} n_i = n \tag{2.22}$$

such that $n_i \leqslant k_i$ for all i.

Note here that (2.21) is obtained for $\mathcal{C}_d$ by considering classical strategies. The detailed proof is given in Appendix B. For a particular case of *Result* 3 wherein $d_y = \overline{d} = d$ for all y, the proof is previously given in [88]. Hence, when $d \leqslant \min_y d_y$, a necessary criterion for a set of measurements to be compatible is given by

$$S(n,\vec{d},d) \leqslant S_c(n,\vec{d},d), \tag{2.23}$$

where $S_c(n,\vec{d},d)$ is given by (2.21).

For $n = 2$, $d_y = \overline{d}$ for all y, and $d \leqslant \overline{d}$, the expression (2.21) simplifies to (for details, see Appendix C)

$$S_c(2,\overline{d},d) = \frac{1}{2\overline{d}^2}\left(d + 2d\overline{d} - d^2\right). \tag{2.24}$$

And for $n = 3$, $d_y = \overline{d}$ for all y, and $d \leqslant \overline{d}$, the expression (2.21) simplifies to (for details, see Appendix C)

$$S_c(3,\overline{d},d) = \frac{d}{3\overline{d}^3}\left(d^2 - 1 + 3\overline{d}(\overline{d} + 1 - d)\right). \tag{2.25}$$

The particular case of *Result* 2 for $n = 2$ can be found in [42], and, moreover, it is shown that any pair of rank-one projective measurements that is incompatible provides advantages in RAC [61]. In order to showcase the generic applicability of *Results* 2 and 3, we consider an arbitrary set of three rank-one projective qubit measurements, which using some unitary can be expressed as

$$\begin{aligned}
M_{x_1|1} &= (1/2)U\left[\mathbb{1} + (-1)^{x_1}\sigma_z\right]U^\dagger \\
M_{x_2|2} &= (1/2)U\left[\mathbb{1} + (-1)^{x_2}\left(\alpha\sigma_z + \sqrt{1-\alpha^2}\sigma_x\right)\right]U^\dagger \\
M_{x_3|3} &= (1/2)U\Big[\mathbb{1} + (-1)^{x_3}(\beta\sigma_z + \gamma\sqrt{1-\beta^2}\sigma_x \\
&\qquad\qquad \pm\sqrt{1-\beta^2}\sqrt{1-\gamma^2}\sigma_y)\Big]U^\dagger
\end{aligned} \tag{2.26}$$

where $x_1, x_2, x_3 \in [2]$, the variables $\alpha, \beta, \gamma \in [-1,1]$, and U can be an arbitrary unitary operator acting on $\mathbb{C}^2$. We obtain the following result.

Result 4. *The figure of merit* (2.16) *of RACs for* $n = 3, \overline{d} = 2, d = 2$ *can witness any set of three incompatible rank-one projective qubit measurements, except for the sets defined by (2.26) with*

$$(\alpha,\beta,\gamma) = \{(\pm 1/2, \pm 1/2, -1), (\pm 1/2, \mp 1/2, 1)\}.$$

This result is proved with the help of numerical optimizations, and the proof is available in Appendix D.

2.4 Generic relations between probability sets

We now point out a few generic relations between the sets in order to get a generic perspective. It is trivial that $\overline{\mathcal{Q}}_d \not\subset \mathcal{C}_d$ since we observe the quantum advantage for $S(n,\vec{d},d)$. We also observe the following,

Fact 2. *In general, $\mathcal{C}_d \not\subset \overline{\mathcal{Q}}_d$, and thus, $\overline{\mathcal{Q}}_d^C \subsetneq \mathcal{C}_d$. In other words, there exist probabilities that belong to $\mathcal{C}_d$ but do not belong to $\overline{\mathcal{Q}}_d$. Moreover, in general, $\mathcal{Q}_d \setminus (\mathcal{C}_d \cup \overline{\mathcal{Q}}_d) \neq \varnothing$. In other words, there exists a probability distribution that belongs to $\mathcal{Q}_d$ but does not belong to $\mathcal{C}_d$ and $\overline{\mathcal{Q}}_d$.*

Proof. Once again, reckon the RAC task and, instead of the average success probability (2.16), let us consider the figure of merit to be the worst case success probability,

$$W(n,\tilde{d},d) = \min_{x,y}\{p(b_y = x_y|x,y)\}. \tag{2.27}$$

It follows from Yao's principle that the average success probability $S(n,2,2)$ is the same as $W(n,2,2)$ when pre-shared randomness is available. See Lemma 1 in [41] for the proof. Therefore, $W(4,2,2)$ in $\mathcal{C}_2$ is the same as $S_c(4,2,2) = 11/16$ [41]. However, it was proven that the best value of $W(4,2,2)$ in $\overline{\mathcal{Q}}_2$ is 1/2 [90], implying $\mathcal{C}_d \not\subset \overline{\mathcal{Q}}_d$. Moreover, since $\overline{\mathcal{Q}}_d^C$ is a subset of both $\mathcal{C}_d$ and $\overline{\mathcal{Q}}_d$, it must be a proper subset.

Besides, there exists a quantum strategy in $\mathcal{Q}_2$ that achieves $W(4,2,2) = 0.74$ (section 4.1.2 in [41]). This means that both $\mathcal{C}_d$ and $\overline{\mathcal{Q}}_d$ are proper subsets of $\mathcal{Q}_d$. □

Next, we present another observation that helps in completely understanding the relationship between different sets of probability distributions.

Fact 3. *In general, $(\mathcal{C}_d \cap \overline{\mathcal{Q}}_d) \setminus \overline{\mathcal{Q}}_d^C \neq \varnothing$. In other words, there exists a set of probabilities that does not belong to $\overline{\mathcal{Q}}_d^C$ at the same time inside both of the sets $\mathcal{C}_d$ and $\overline{\mathcal{Q}}_d$.*

Proof. In $\mathcal{C}_d$, $W(3,2,2) = S_c(3,2,2) = 3/4$ [41]. It is also well known that there exist two-dimensional quantum states and measurements that lead to $W(3,2,2) = 1/2 + 1/(2\sqrt{3}) \approx 0.79$, without pre-shared randomness. An example of such quantum states is given in *Example 2* of Ref. [90]. If we consider the noisy version of

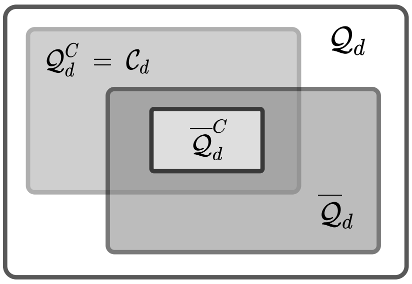

FIG. 2.2 Generic relation between different sets of probabilities. The equivalence between $\mathcal{Q}_d^C$ and $\mathcal{C}_d$ is shown in *Result 1*. There exist many examples such that $\overline{\mathcal{Q}}_d$ is not a subset of $\mathcal{C}_d$. On the other hand, Fact 2 points out that $\mathcal{C}_d$ is not a subset of $\overline{\mathcal{Q}}_d$ and both $\mathcal{C}_d$ as well as $\overline{\mathcal{Q}}_d$ are proper subsets of $\mathcal{Q}_d$. While it is immediate that $\overline{\mathcal{Q}}_d^C$ is a subset of both $\mathcal{C}_d$ and $\overline{\mathcal{Q}}_d$, Fact 3 shows that $\overline{\mathcal{Q}}_d^C$ is not the intersection of these two sets.

those quantum states $\nu\rho + (1-\nu)\mathbb{1}/2$ such that $\nu = \sqrt{3}/2$, then it is easy to see that the value of $W(3,2,2)$ reduces to 3/4. Therefore, $W(3,2,2) = 3/4$ is obtained in both of the sets $\mathcal{C}_2$ and $\overline{\mathcal{Q}}_2$. On the other hand, a numerical analysis using semi-definite programming shows that $W(3,2,2)$ is, at most, 2/3 in $\overline{\mathcal{Q}}_2^C$, which implies $(\mathcal{C}_2 \cap \overline{\mathcal{Q}}_2) \setminus \overline{\mathcal{Q}}_2^C \neq \varnothing$. □

Based on the above observations, the generic relationship between different sets of probabilities can be understood as depicted in FIG. 2.2.

2.5 Concluding remarks

By characterizing the set of quantum correlations in prepare-and-measure scenarios produced from any set of compatible measurements, we have shown here that incompatible measurements at the receiver's end are necessary to demonstrate a quantum advantage in any communication task. Based on this result, we have presented a semi-device-independent witness of measurement incompatibility invoking generalized random access codes. Further, we have completely characterized the sets of three incompatible projective qubit measurements that can be detected using our proposed witness. The relationship between the classical probability distributions and different types of quantum probability distributions produced in an arbitrary communication task has also been presented.

It might be noted that some of the results derived in [42, 61] appear as natural corollaries of the results obtained here. It has recently been shown that the measurement statistics produced in any communication task involving compatible measurements by the receiver can be reproduced by classical communication, where the dimension of the classical communication is of the order of the number of outcomes of the parent POVM [79]. Significantly, our results further reduce the dimension of the classical communication involved.

The importance of the analysis presented here lies in the fact that the classical bound of the success metric of any communication task becomes an upper bound on the metric of the task under a compatible set of measurements. Consequently, violating the classical bound of any communication task can be used as a sufficient criterion to witness measurement incompatibility. These bounds are tight whenever the dimension on which the measurements act is not larger than the number of outcomes of any of the measurements. The present study establishes that measurement incompatibility is the fundamental quantum resource for non-classicality in any communication task or, more generally, in prepare-and-measure scenarios.

Our analysis paves the way for investigations of several open questions. First, deriving more efficient incompatibility witnesses based on different communication tasks is worthwhile for future studies. Second, our results may be generalized to propose semi-device-independent witnesses for incompatible quantum channels [91] and quantum instruments [92, 93]. Finally, proposing operational witnesses for all incompatible extremal POVMs [94] is another fundamentally motivated open problem.

Chapter 3

An operational approach to classifying measurement incompatibility

3.1 Introduction

The significance of measurement incompatibility in various operational tasks [24, 76, 95, 96] calls for its in-depth characterization. Towards this direction, a classification of measurement incompatibility with respect to projection onto subspaces has been recently performed [97], also different hierarchy of incompatible measurements has been given based on the number of copies of quantum states required [98], and the simulability of their statistics produced in the experiments [99, 100]. In the continuous variable system, the verification of incompatibility using phase-space quasiprobability distributions has been studied in Ref. [101].

In the present work, our objective is to classify measurement incompatibility in an operational approach that does not involve the details of a theory. To put it differently, we address how various degrees of incompatibility can be assessed solely by executing basic classical operations on the inputs or outputs of these measurements. From an operational perspective, when the concerned measurement devices are black boxes with no control over the internal workings, one can still realise different measurements by performing suitable classical operations to manipulate the statistics of the measurements. Here, we employ two such operations: coarse-graining of measurement outcomes (classical post-processing performed on the outcomes of a measurement) and convex-mixing of measurement settings (classical operation performed on the inputs). One can also consider classical operations performed on both inputs and outputs. Similar types of classical operations were also introduced in [102], nevertheless, the study of incompatibility under these classical operations was lacking. The motivation of this present work is to fill this crucial gap in the literature.

Coarse-graining of measurement outcomes arises naturally in several instances, for example, in measurements on continuous variable systems [103]. Though the eigen spectra of the observables are infinite-dimensional and continuous, real-world experimental devices are limited by finite precision, leading to the measurement outcomes taking a finite number of discrete values. This inaccuracy in the recording of measurement outcomes is manifested in the coarse-graining of measurement outcomes, which is inevitable in practice. On the other hand, device imperfection may also lead to the measurement device performing a set of measurements probabilistically, instead of always performing the desired particular measurement. In such a case, a convex-mixing of the given set of measurements arises effectively [104].

By leveraging these classical operations on the measurement device, we can learn the finer details about the measurements i.e. the different layers of incompatibility, which can give a measure of the degree of incompatibility of a set of measurements. This can set a benchmark for legitimately choosing incompatible measurements for an information processing task. Our study is motivated to address how one may compare the degree of incompatibility between two different sets of measurements subjected to the aforementioned classical operations. For instance, if the first set remains incompatible for every possible non-trivial coarse-graining of the measurement outcomes, but the second set becomes compatible for a certain coarse-graining, it follows that the first set of measurements exhibits stronger incompatibility compared to the second one. A similar argument holds for the case of convex-mixing of measurements also.

In this work, we establish analytical criteria for determining when a pair of projective measurements are fully incompatible, i.e., remain incompatible under all possible coarse-grainings of measurement outcomes. We further analyze the full incompatibility of a set of three qubit measurements under all possible convex-mixing as well. Within the context of our present study, noise is reflected in degrading the incompatibility properties of various measurement sets [105]. We compute the critical noise threshold below which the mutually unbiased bases measurements remain incompatible under the above-mentioned classical operations.

As incompatible measurements are useful for various information processing tasks, any device claiming to produce incompatible measurements must be certified before using it in an experiment. Verification of incompatibility of the measurements is possible from the input-output measurement statistics obtained from the device without knowing its internal functioning. This can be done in two ways one is a device-independent way utilizing Bell-type experiments [106, 74, 75] and another is a semi-device-independent way inspired by the standard prepare-and-measure scenario [107, 24, 42]. In the present work, we investigate the issue of certification of different layers of measurement incompatibility under the introduced classical operations both from the device-independent and the semi-device-independent perspective. As both Bell-type experiments and prepare-and-measure experiments [108, 109] are viable with the current technology, by performing our introduced classical operations on those experiments we can operationally certify the different layers of incompatibility thus allowing us to know the subtleties of incompatibility about the measurement device which is a new and interesting offshoot of our work.

This chapter is structured as follows. In Sec.(3.2), the significance of these classical operations in an operational paradigm is discussed along with the physical ground for these classical operations. In Sec. (3.3), we define a hierarchy in the incompatibility of measurements under these classical operations. In Sec. (3.4), we study the noise robustness for various levels of incompatibility of measurements subjected to the above classical operations. In Sec. (3.5), we define operational witnesses of incompatibility of measurements under these classical operations and furnish examples to study their performance in device-independent and semi-device-independent frameworks. Concluding remarks are presented in Sec.(3.6).

3.2 Operational approach to classifying incompatibility

Consider a measurement device where, when a physical system is probed, we can choose measurement settings via a classical parameter, denoted by x. This parameter serves as the input to the device, and the device produces the measurement outcome z, corresponding to that input. Suppose there are n possible inputs, i.e., $x \in [n]$, and for each input, there are d possible outcomes, i.e., $z \in [d]$, where $[k]$ represents the set $\{0, \cdots, k-1\}$ for any natural number k. For simplicity, we assume that the number of outcomes for each measurement is the same. Without loss of generality, this assumption holds, as any measurement with fewer than d outcomes can be treated as having d outcomes by considering some outcomes as never occurring. By probing different system preparations, we can gather input-output statistics from the device.

In quantum mechanics, measurements are described by Positive Operator Valued Measures (POVMs), which consist of positive semi-definite operators that sum up to the identity operator. Let us represent the measurements realized in the device by the set of operators $\{M_{z|x}\}_{z,x}$, where x indexes different measurements, and z denotes the corresponding outcomes. A set of measurements is said to be compatible if there exists a parent POVM, G_λ, and classical post-processing $\{p(z|x,\lambda)\}$ for each x such that

$$\forall z,x, \quad M_{z|x} = \sum_{\lambda} p(z|x,\lambda) G_\lambda, \tag{3.1}$$

where $0 \leq p(z|x,\lambda) \leq 1$, and $\sum_z p(z|x,\lambda) = 1$, for all x, λ [29]. As a special case, if the operators are projectors, then the two measurements are jointly measurable when their corresponding operators commute.

In an operational paradigm, we do not have direct control over the internal workings of the measurement device. However, we can manipulate the classical inputs and outputs to realize different measurements.

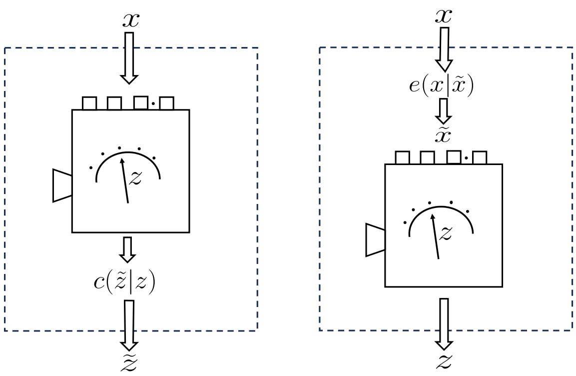

FIG. 3.1 Classical operations on the outputs (left) and classical operations on the inputs (right) to realize a new set of measurements are depicted.

3.2.1 Classical operations on outputs

In general, one can modify the outcomes of a measurement by applying a post-processing operation or stochastic map (see FIG. 3.1). For a given input x, this operation is mathematically described by a probability distribution $\mathcal{C}_x := \{c_x(\tilde{z}_x|z)\}$, $c_x(\tilde{z}_x|z) \geqslant 0$ and $\sum_{\tilde{z}_x} c_x(\tilde{z}_x|z) = 1$. Here $\tilde{z}_x \in [\tilde{d}_x]$ denotes the outcome of the new measurement after the operation, and $\mathcal{C}_x$ is the stochastic map with $c_x(\tilde{z}_x|z)$ is the probability of producing outcome $\tilde{z}_x$ when outcome z is obtained for the measurement x. In general, one can apply different operations for different

x. By applying different operations for each x, one can generate a new set of measurements described by the operators $\{\tilde{M}_{\tilde{z}_x|x}\}_{\tilde{z}_x,x}$ as follows:

$$\tilde{M}_{\tilde{z}_x|x} = \sum_z c_x(\tilde{z}_x|z) M_{z|x} \,, \tag{3.2}$$

for each x. If the initial set of measurements is incompatible, we can ask whether this new set is incompatible or not.

The set of all possible operations $\{\mathcal{C}_x\}$ is infinite. Furthermore, certain operations, such as generating random outputs, will always result in a set of measurements that is compatible, regardless of the original measurements. To avoid such intricacies, we focus on a particular class of operations where $c_x(\tilde{z}_x|z) \in \{0,1\}$ and $\tilde{d}_x < d$. These classical operations involve coarse-graining or relabelling/permuting outcomes (or a combination of both). This class contains a finite number of operations, and interestingly, any classical operation on outputs can be expressed as convex combinations of this class of operations. More importantly, certain quantum measurements remain incompatible even after all possible operations of this kind. We will refer to this class of operations as *coarse-graining*. Coarse-graining typically refers to treating more than one outcome as equivalent, *i.e.*, multiple outcomes are clubbed together to form a single outcome. As a result, the effective number of outcomes is reduced.

3.2.2 Classical operation on inputs

We can also apply classical operations to the inputs to generate a new set of measurements, each of which is a convex combination of the initial measurements (see FIG. 3.1). This classical operation is represented by a probability distribution $\{e(x|\tilde{x})\}_{x,\tilde{x}}$, where $e(x|\tilde{x}) \geqslant 0$ and $\sum_x e(x|\tilde{x}) = 1$ for each $\tilde{x} \in [m]$. Here, $e(x|\tilde{x})$ refers to the convex weightage of appearing measurement x in the new measurement labelled by $\tilde{x}$. The new set of m measurements, defined by the operators $\{\tilde{M}_{z|\tilde{x}}\}_{z,\tilde{x}}$, is given by:

$$\tilde{M}_{z|\tilde{x}} = \sum_x e(x|\tilde{x}) M_{z|x}, \tag{3.3}$$

for every z.

If the same convex mixing is applied uniformly to all inputs, i.e., $e(x|\tilde{x})$ is the same for all $\tilde{x}$, the resulting measurements will be identical and hence compatible. To avoid such cases, we introduce a particular class of convex mixing, which we will refer to as *disjoint-convex-mixing* of measurements. In disjoint-convex-mixing, no initial measurement appears in the convex combination of more than one new measurement. Mathematically, for every z, only one of the numbers from the set $\{e(x|\tilde{x})\}_{\tilde{x}}$ is non-zero, and others are zero, which can be expressed as:

$$\forall x,\ \max_{\tilde{x}}\{e(x|\tilde{x})\} = \sum_{\tilde{x}} e(x|\tilde{x}). \tag{3.4}$$

In this work, we focus on disjoint-convex-mixing for classifying measurement incompatibility. We will point out later that there are measurements that remain incompatible after all possible disjoint-convex-mixing.

3.2.3 Classical operations on both inputs and outputs

It is also possible to construct new measurements by combining operations on both inputs and outputs. This can be done in two ways - first applying an operation on the inputs followed by the outputs, or vice versa.

In the former case, the operators $\{\tilde{M}_{\tilde{z}_{\tilde{x}}|\tilde{x}}\}_{\tilde{z}_{\tilde{x}},\tilde{x}}$ defining the new measurements are

$$\tilde{M}_{\tilde{z}_{\tilde{x}}|\tilde{x}} = \sum_z \sum_x c(\tilde{z}_x|z) e(x|\tilde{x}) M_{z|x}\ , \tag{3.5}$$

where $\{e(x|\tilde{x})\}$ and $\{c(\tilde{z}_x|z)\}_x$ denote the classical operation on inputs and outputs, respectively. In the latter case, the set of operators $\{\tilde{M}_{\tilde{z}|\tilde{x}}\}$ are

$$\tilde{M}_{\tilde{z}|\tilde{x}} = \sum_x \sum_z e(x|\tilde{x}) c(\tilde{z}|z) M_{z|x}\ , \tag{3.6}$$

where $\{e(x|\tilde{x})\}$ and $\{c(\tilde{z}|z)\}$ denote the classical operation on inputs and outputs, respectively. Here again, without loss of generality, we consider the number of outcomes after the operations on the outcomes to be the same, which is denoted by $\tilde{z}$.

It is important to note that these two cases are not equivalent, as the stochastic operations on the outputs apply to different sets of measurements in each case.

In summary, one can explore the effects of various classical operations on both inputs and outputs to determine whether the resulting measurements remain incompatible. However, this work specifically focuses on two key classes of operations: coarse-graining of outputs and disjoint-convex-mixing of inputs. In addition to the operational relevance of the two types of operations mentioned above, their significance for practical implementation is discussed in the following subsection.

3.2.4 Physical motivations for considering the two specific classical operations

Let us now elaborate further the motivations for considering the two aforementioned specific classical operations on a set of measurements. Coarse-graining of measurement outcomes is a natural consequence of an observer's limitation in a practical scenario involving multi-outcome measurements. For example, consider the measurement of the spin-z component (associated with the operator J_z) of the spin-j system, where j is very large. The possible outcomes of this measurement are nothing but the eigenvalues of J_z, denoted by $m \in \{-j, -j+1, \cdots, j\}$. In practice, this type of measurement is performed using a Stern-Gerlach type experiment. In such a practical situation, the concept of "neighbouring outcomes" arises [110]. For example, the outcomes m and $m+1$ are neighbouring outcomes in the real configuration space in a Stern-Gerlach type experiment. For large j, it is almost impossible for a device with finite precision to resolve these neighbouring outcomes in the observation screen – giving rise to coarse-graining of measurement outcomes. This type of practical limitation is not only limited to large spins but also applicable to other multi-outcome measurements and measurements of continuous variables (e.g., position, momentum [111]). Note that coarse-graining of measurement outcomes has been invoked in the context of explaining classical limits of quantum mechanics [110, 112–116].

On the other hand, convex mixing of measurements is also inevitable in a practical situation. To explain it in more detail, let us consider the example of measuring σ_z (Pauli spin observable) on a qubit. In practice, the observable to be measured (σ_z in the present example), defined by the relative direction of the inhomogeneous magnetic field in the Stern-Gerlach apparatus with respect to the direction of the incoming beam of spin-$1/2$ particles, may not be kept fixed in all experimental runs. Consequently, instead of σ_z, $\vec{\sigma}.\hat{n}$ will be measured ($\vec{\sigma}.\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$ with σ_x, σ_y, σ_z are three Pauli operators and $n_x^2 + n_y^2 + n_z^2 = 1$), where $\hat{n}$ will be different in different runs (all such $\hat{n}$ should be close to $\hat{z}$). Effectively, this will give rise to convex mixing of different measurements.

Hence, these operations are extremely relevant in practical scenarios. The various layers of incompatibility and their operational implications under these operations are discussed in the subsequent sections.

3.3 Classifying measurement incompatibility under coarse-graining and disjoint-convex-mixing

3.3.1 Coarse-graining of outcomes

Recall that the coarse-graining of a set of d-outcome measurements $\{M_{z|x}\}$ produces new measurements defined by

$$\tilde{M}_{\tilde{z}_x|x} = \sum_z c_x(\tilde{z}_x|z) M_{z|x}, \text{ where } c_x(\tilde{z}_x|z) \in \{0,1\}. \tag{3.7}$$

Before moving forward, let us define what we mean by *trivial coarse-graining*. If the coarse-graining results in a measurement where one of the outcomes, say $\tilde{z}_x$, always occurs, that is, $\tilde{M}_{\tilde{z}_x|x} = \mathbb{1}$, we refer that operation as *trivial coarse-graining* for the measurement labelled by x. The fact that two incompatible measurements may not necessarily remain incompatible after certain coarse-graining motivates the following definition of full incompatibility with respect to (w.r.t.) coarse-graining.

Definition 1 (Fully incompatible measurements w.r.t. coarse-graining)**.** *A set of measurements $\{M_{z|x}\}$ is fully incompatible w.r.t. coarse-graining if they remain incompatible after all possible nontrivial coarse-graining. That is if the resultant set of measurements given by Eq.* (3.7) *after all possible sets of nontrivial coarse-graining is incompatible, then we call them fully incompatible. Note that, the coarse-graining can be different for different settings x.*

Definition 2 (k-incompatible measurements w.r.t. coarse-graining)**.** *A set of measurements $\{M_{z|x}\}$ is k-incompatible w.r.t. coarse-graining if they remain incompatible after all possible nontrivial coarse-graining that gives rise to at least k outcome measurements. In other words, if the resultant set of measurements given by Eq.* (3.7)*, where $\tilde{z}_x \in [d_x]$ and $d_x \geqslant k$ for all x, is incompatible, then we call them k-incompatible.*

For example, the three outcome rank-one projective measurement pair, defined by the following (un-normalized) vectors

$$M = \{|0\rangle, |1\rangle, |2\rangle\} \tag{3.8}$$

and

$$N = \{\, |0\rangle + |1\rangle, |0\rangle - |1\rangle, |2\rangle \,\} \tag{3.9}$$

is 3-incompatible, but not 2-incompatible since coarse-graining of the first two outcomes of these measurements yields compatible measurements.

Result 5. *A set of fully incompatible measurements is equivalent to 2-incompatible measurements w.r.t. coarse-graining.*

Proof. If a set of measurements is 2-incompatible, then it implies that the set remains incompatible after all possible coarse-graining of the outcomes such that the number of outcomes of each measurement in the newly formed set of measurements is greater than or equal to two. Also, the lowest number of outcomes of measurement is two for a nontrivial coarse-graining. Furthermore, if a set of measurements is d-incompatible, then, by definition, it is n-incompatible as well, where $n > d$, but the converse is not true. This proves Observation 1. □

Result 6. *Consider two projective measurements, defined by* $\{P_i\}$, *and* $\{Q_j\}$, *where* $i \in [d]$ *and* $j \in [d']$. *Let* $\{\mathcal{M}_k\}_k$ *be the set of all proper subsets of* $[d]$, *and* $\{\mathcal{N}_l\}_l$ *be the set of all proper subsets of* $[d']$. *Then these two measurements are fully incompatible w.r.t. coarse-graining if and only if*

$$\left[\sum_{i\in\mathcal{M}_k} P_i, \sum_{j\in\mathcal{N}_l} Q_j\right] \neq 0, \ \forall k,l. \tag{3.10}$$

Proof. The result is a direct consequence of the fact that for sharp measurement, compatibility and commutativity are equivalent [117]. Suppose $\exists k,l$, such that the left-hand-side of (3.10) is zero. Then consider the coarse-graining such that the resulting measurements will be $\{\sum_{i\in\mathcal{M}_k} P_i, \mathbb{1} - \sum_{i\in\mathcal{M}_k} P_i\}$ and $\{\sum_{j\in\mathcal{N}_l} Q_j, \mathbb{1} - \sum_{j\in\mathcal{N}_l} Q_j\}$. The resultant measurements will be compatible. The converse direction holds true from the definition of fully incompatible w.r.t. coarse-graining. □

Theorem 1. *The following condition is necessary but not sufficient for two rank-one projective measurements of dimension* ≥ 4, *defined by* $\{|\psi_i\rangle\}$ *and* $\{|\phi_j\rangle\}$, *to be fully incompatible w.r.t. coarse-graining:*

$$\langle\psi_i|\phi_j\rangle \neq 0, \ \forall i,j. \tag{3.11}$$

However, the above condition is necessary and sufficient for two 3-dimensional rank-one projective measurements.

Proof. First, note that for sharp measurement, compatibility and commutativity are equivalent [117]. Suppose $\exists i,j$, such that $\langle\psi_i|\phi_j\rangle = 0$. Consider coarse-graining of all other outcomes except i and j for the two measurements. Then, the resulting measurements will be $\{|\psi_i\rangle\langle\psi_i|, \mathbb{1} - |\psi_i\rangle\langle\psi_i|\}$ and $\{|\phi_j\rangle\langle\phi_j|, \mathbb{1} - |\phi_j\rangle\langle\phi_j|\}$, which commute with each other (i.e., the resulting measurements are compatible) since $\langle\psi_i|\phi_j\rangle = 0$. Thus, if the measurements are fully incompatible, condition (3.11) holds.

To show that (3.11) is not sufficient, consider the following two 4-dimensional rank-one projective measurements (with the normalization factor $1/\sqrt{2}$),

$$\begin{aligned} &\{|0\rangle + |1\rangle, |0\rangle - |1\rangle, |2\rangle + |3\rangle, |2\rangle - |3\rangle\} \\ &\{|0\rangle + |2\rangle, |0\rangle - |2\rangle, |1\rangle + |3\rangle, |1\rangle - |3\rangle\}. \end{aligned} \tag{3.12}$$

One can check that (3.11) holds for all $i, j = 0,1,2,3$. But coarse-graining of outcomes $0,1$ and $2,3$ for both the measurements leads to

$$\begin{aligned} &\{|0\rangle\langle 0| + |1\rangle\langle 1|, |2\rangle\langle 2| + |3\rangle\langle 3|\}, \\ &\{|0\rangle\langle 0| + |2\rangle\langle 2|, |1\rangle\langle 1| + |3\rangle\langle 3|\}, \end{aligned} \tag{3.13}$$

which are compatible.

In 3-dimension, say, the measurements are $M = \{|\psi_i\rangle\}$ and $N = \{|\phi_j\rangle\}$ with $i, j \in \{1,2,3\}$. Now any non-trivial coarse-graining yields binary-outcome measurements of the form: $\{|\psi_i\rangle\langle\psi_i|, \mathbb{1} - |\psi_i\rangle\langle\psi_i|\}$ and $\{|\phi_j\rangle\langle\phi_j|, \mathbb{1} - |\phi_j\rangle\langle\phi_j|\}$. It is easy to see that these two remain incompatible if and only if $[|\psi_i\rangle\langle\psi_i|, |\phi_j\rangle\langle\phi_j|] \neq 0$, which is equivalent to $\langle\psi_i|\phi_j\rangle \neq 0,1$. In the case where $\langle\psi_i|\phi_j\rangle = 1$, there exists another pair (i, j') such that $\langle\psi_i|\phi_{j'}\rangle = 0$; thus, (3.11) implies fully incompatible in dimension 3. □

3.3.2 Disjoint-convex-mixing of measurements

Consider a device that implements three different measurements, $M = \{M_z\}_z, N = \{N_z\}_z$, and $R = \{R_z\}_z$, all having the same number of outcomes $z \in [d]$. For the first input, it performs measurement M with probability q and N with probability $(1-q)$. For the second input, it always performs measurement R. Thus, the new measurement $Q_{(M,N)}$, realized through the convex-mixing, is

$$Q_{(M,N)} = \{qM_z + (1-q)N_z\}_z, \tag{3.14}$$

where $q \in [0,1]$ is the weightage of the disjoint-convex-mixing. Even if R is incompatible with M and N separately, R is not necessarily incompatible with

$Q_{(M,N)}$ for all values of q. In a similar way, we introduce the notion of full incompatibility w.r.t. disjoint-convex-mixing.

Definition 3. *Three measurements M, N and R are fully incompatible w.r.t. disjoint-convex-mixing if each of the pairs, M and $Q_{(N,R)}$, N and $Q_{(M,R)}$, R and $Q_{(M,N)}$, are incompatible for all values of q, where $Q_{(\, , \,)}$ is defined in* (3.14).

Consider three unbiased qubit binary outcome measurements $\{M_0, M_1\}, \{N_0, N_1\}, \{R_0, R_1\}$,

$$\begin{aligned} M_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \, \vec{n}_0 \cdot \vec{\sigma}\right), \\ N_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \, \vec{n}_1 \cdot \vec{\sigma}\right), \\ R_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \, \vec{n}_2 \cdot \vec{\sigma}\right), \end{aligned} \tag{3.15}$$

with $z = 0,1$ and $||\vec{n}_i|| \leqslant 1$ where $i \in \{0,1,2\}$. A necessary and sufficient criterion for the incompatibility of two unbiased binary-outcome qubit measurements is given by Busch in [28] (also, see eq.(7) of [29]). By applying this criterion, we find that the above three measurements (3.15) are fully incompatible w.r.t. disjoint-convex-mixing if and only if,

$$||\vec{n}_i + q\vec{n}_j + (1-q)\vec{n}_k|| + ||\vec{n}_i - q\vec{n}_j - (1-q)\vec{n}_k|| > 2, \tag{3.16}$$

for all q, and for all $(i,j,k) \in \{(0,1,2),(1,2,0),(2,0,1)\}$.

Theorem 2. *If three-qubit measurements* (3.15) *are such that $\vec{n}_i$ are in the same plane of the Bloch sphere, then they are not fully incompatible w.r.t. disjoint-convex-mixing.*

Proof. If the three $\vec{n}_i$ are in the same plane, then there exists at least one triple (i,j,k) such that

$$\vec{n}_0 = \frac{1}{c}(q\vec{n}_1 + (1-q)\vec{n}_2) \tag{3.17}$$

for some $c \in (0,1], q \in [0,1]$. In other words, there exists a triple (i,j,k) so that $\vec{n}_i$ is expressed as a linear combination of $\vec{n}_j$ and $\vec{n}_k$ with non-negative coefficients (here q/c and $(1-q)/c$), where the sum of those two non-negative coefficients is greater than or equal to 1. Substituting this into left hand side of (3.16), we find

$$||(1+c)\vec{n}_i|| + ||(1-c)\vec{n}_i|| \leqslant |1+c| + |1-c| = 2. \tag{3.18}$$

This contradicts with (3.16), implying they are compatible. □

Theorem 3. *Consider three-qubit measurements* (3.15) *are such that* $\vec{n}_0 = \nu_0 \hat{x}$, $\vec{n}_1 = \nu_1 \hat{y}$, $\vec{n}_2 = \nu_2 \hat{z}$ *with* $0 \leqslant \nu_0, \nu_1, \nu_2 \leqslant 1$, *that is, the noisy version of Pauli observables,*

$$
\begin{aligned}
M_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_0 \sigma_x\right) = \nu_0 \left(\tfrac{\mathbb{1}+(-1)^z\sigma_x}{2}\right) + (1-\nu_0)\tfrac{\mathbb{1}}{2}, \\
N_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_1 \sigma_y\right) = \nu_1 \left(\tfrac{\mathbb{1}+(-1)^z\sigma_y}{2}\right) + (1-\nu_1)\tfrac{\mathbb{1}}{2}, \\
R_z &= \tfrac{1}{2}\left(\mathbb{1} + (-1)^z \nu_2 \sigma_z\right) = \nu_2 \left(\tfrac{\mathbb{1}+(-1)^z\sigma_z}{2}\right) + (1-\nu_2)\tfrac{\mathbb{1}}{2},
\end{aligned}
\tag{3.19}
$$

with $z = 0,1$. *These measurements are fully incompatible w.r.t. disjoint-convex-mixing if and only if*

$$
\min\left\{ \nu_0^2 + \frac{\nu_1^2\nu_2^2}{\nu_1^2+\nu_2^2}, \nu_1^2 + \frac{\nu_0^2\nu_2^2}{\nu_0^2+\nu_2^2}, \nu_2^2 + \frac{\nu_0^2\nu_1^2}{\nu_0^2+\nu_1^2} \right\} > 1. \tag{3.20}
$$

Proof. In terms of ν_i, the left hand side of (3.16) becomes $2\sqrt{\nu_i^2 + q^2\nu_j^2 + (1-q)^2\nu_k^2}$. Note that the minimum of $q^2\nu_j^2 + (1-q)^2\nu_k^2$ occurs at $\tilde{q} = \nu_k^2/(\nu_j^2+\nu_k^2)$. Since $0 \leq \nu_k^2/(\nu_j^2+\nu_k^2) \leq 1$ for any $\nu_j, \nu_k \in [0,1]$, the above-mentioned minimum can always be achieved with a suitable choice of q. Hence, we have that

$$
\begin{aligned}
2\sqrt{\nu_i^2 + q^2\nu_j^2 + (1-q)^2\nu_k^2} &\geq 2\sqrt{\nu_i^2 + \tilde{q}^2\nu_j^2 + (1-\tilde{q})^2\nu_k^2} \\
&= 2\sqrt{\nu_i^2 + \frac{\nu_j^2\nu_k^2}{\nu_j^2+\nu_k^2}}
\end{aligned}
\tag{3.21}
$$

Thus, the right-land-side of (3.16) is greater than 2 for all values of q if and only if

$$
\nu_i^2 + \frac{\nu_j^2\nu_k^2}{\nu_j^2+\nu_k^2} > 1. \tag{3.22}
$$

Taking all the three possible values of (i,j,k) we get the condition (3.20). □

Clearly, the three Pauli observables are fully incompatible w.r.t. disjoint-convex-mixing, and moreover, if we take an equal amount of noise $\nu_0 = \nu_1 = \nu_2 = \nu$, then (3.20) implies $\nu > \sqrt{2/3}$.

We can generalize the notion of full incompatibility w.r.t. disjoint-convex-mixing.

Definition 4 (k-incompatible measurements w.r.t. disjoint-convex-mixing)**.** *Given a set of n measurements, the measurements are k-incompatible w.r.t. disjoint-convex-mixing if, after taking every possible disjoint-convex-mixing that yields k number of measurements, the resulting measurements are incompatible.*

Definition 5 (Fully incompatible measurements w.r.t. disjoint-convex-mixing)**.** *A set of n measurements is fully incompatible w.r.t. disjoint-convex-mixing if it is k-incompatible for all $k = 2, \cdots, n$.*

Result 7. *Fully incompatible measurements w.r.t. disjoint-convex-mixing imply that every pair of measurements from that set is incompatible. The reverse implication does not hold.*

Proof. Consider a set of n measurements, in which there is a pair of measurements $\{M_z\}$ and $\{N_z\}$ that are compatible. Now if we make two partitions where $\{M_z\}$ and $\{N_z\}$ are in different partitions and the disjoint-convex-mixing is such that the probabilities of arising all other measurements are zero, then the resultant measurement pair must be compatible. This is true for any compatible pair of measurements. Thus, if the measurements are fully incompatible w.r.t. disjoint-convex-mixing, every pair of measurements must necessarily be incompatible.

The converse is not true. Consider the three noisy Pauli measurements of Eq.(3.19). It can be shown by using semi-definite programming that if $0.71 < \nu \leqslant 0.81$, the measurements are pairwise incompatible, but it is not fully incompatible w.r.t. disjoint-convex-mixing [118]. □

Some explicit examples of k-incompatible measurements w.r.t. both of these operations are provided in the subsequent section.

3.4 Robustness under noise for different levels of incompatibility

In this section, we analyze the role of noise on quantum measurements and study how the incompatibility properties depend on it. More specifically, we study the variation of the critical amount of noise in the different layers of incompatibility under these classical operations (coarse-graining and disjoint-convex-mixing). Due to the ubiquitous nature of noise, it is pertinent to study the extent to which noise could be tolerated by a set of measurements while still retaining their incompatibility. We take a noisy version of mutually unbiased bases measurements (MUBs),

$$M_{i|x} = \nu \left|\phi_{i|x}\right\rangle \left\langle\phi_{i|x}\right| + \frac{(1-\nu)}{d}\mathbb{1}_d, \tag{3.23}$$

where $\{|\phi_{i|x}\rangle\}_{i,x}$ form mutually unbiased bases measurements in $\mathbb{C}^d$. Here ν is the robustness parameter (or visibility parameter) and $(1-\nu)$ is the noise parameter, $0 \leq \nu \leq 1$. When the noise parameter is zero (i.e., robustness, $\nu = 1$), the measurements are fully incompatible, and when the noise parameter is one (i.e., the robustness, $\nu = 0$), the measurements are trivial and compatible. Our aim is to obtain the critical value of the robustness parameter above which the measurements remain incompatible before and after different classical operations.

To check the compatibility, i.e., the existence of a parent POVM, we use the method described in [29]. This can be cast as a semi-definite programming (SDP) problem that takes a set of measurements $\{M_{z_x|x}\}$ and deterministic classical post-processings $p(z_x|x,\lambda)$ as input, and checks whether the measurements are compatible or not, subject to the constraints

$$\textstyle\sum_\lambda p(z_x|x,\lambda)G_\lambda = M_{z_x|x} \forall x, z_x, \tag{3.24}$$

$$\textstyle\sum_\lambda G_\lambda = \mathbb{1}, \tag{3.25}$$

$$G_\lambda \geq \mu\mathbb{1}, \tag{3.26}$$

where μ is the optimization parameter. This method finds the maximum value of μ for each $\{p(z_x|x,\lambda)\}$. If this optimization returns a negative value of μ, then the constraint of Eq.(3.26) cannot be fulfilled, which implies that the measurements $\{M_{z_x|x}\}$ are incompatible. Otherwise, they are compatible.

As discussed earlier, the degree of incompatibility of measurements may be reduced due to coarse-graining of outcomes and convex mixing of measurements. In the subsequent subsection, we present an explicit analysis of how the degree of incompatibility (quantified by the robustness parameter [119]) varies as the measurements are subjected to coarse-graining and disjoint-convex-mixing, in dimensions 3 and 4. For simplicity, we have considered the measurements to be MUBs (as discussed earlier), where the number of outcomes are same as that of their dimension. Also, as MUBs exhibit the highest degree of incompatibility, [120] it is a natural choice to use them for the prominent observation of the various layers of incompatibility under the classical operations. However, one can apply this formalism to any set of measurements in a similar fashion.

3.4.1 Robustness of incompatibility under coarse-graining of outcomes

Dimension 3. Let $\{M_i\}$ and $\{N_j\}, i, j \in \{1,2,3\}$ be two three-outcome measurements acting on $\mathbb{C}^3$, where

$$\begin{aligned} M_i &= \nu \left|i\right\rangle\left\langle i\right| + (1-\nu)\frac{\mathbb{1}}{3}, \\ N_j &= \nu \left|\psi_j\right\rangle\left\langle \psi_j\right| + (1-\nu)\frac{\mathbb{1}}{3}, \end{aligned} \tag{3.27}$$

with

$$\begin{aligned} \left|\psi_0\right\rangle &= \frac{1}{\sqrt{3}}(\left|0\right\rangle + \left|1\right\rangle + \left|2\right\rangle), \\ \left|\psi_1\right\rangle &= \frac{1}{\sqrt{3}}(\left|0\right\rangle + \omega\left|1\right\rangle + \omega^2\left|2\right\rangle), \\ \left|\psi_2\right\rangle &= \frac{1}{\sqrt{3}}(\left|0\right\rangle + \omega^2\left|1\right\rangle + \omega\left|2\right\rangle), \end{aligned} \tag{3.28}$$

ω being the cube roots of unity. It can be easily checked by SDP that the measurements in Eqs.(3.27) and (3.28) are incompatible (or equivalently, 3-incompatible using the definition 2) for robustness parameter $(\nu) > 0.683$. Now, any non-trivial coarse-graining reduces the measurement outcomes to two, so one can check the incompatibility after all possible choices of such non-trivial coarse-grainings. The measurements are 2-incompatible (by the definition 2) for $\nu > 0.711$. For $0.683 < \nu \leq 0.711$, the measurements are 3-incompatible, but not 2-incompatible. This represents the case where the initial measurements, although taken to be incompatible, become compatible after a non-trivial coarse-graining. However, for $\nu > 0.711$, the measurements are 2-incompatible (and hence 3-incompatible), suggesting that they remain incompatible after all non-trivial coarse-grainings. These findings suggest that 2-incompatible measurements exhibit a stronger form of incompatibility compared to the 3-incompatible measurements (hence supporting Observation 5), and hence are potential candidates for any resource-theoretic applications that harness incompatibility. Note that for three outcome measurements, any non-trivial coarse-graining will reduce the outcomes to two. So, the number of outcomes is always ≤ 3 (where 3 corresponds to the case where there is no coarse-graining). So, 4-incompatibility is not applicable (N.A.) here.

Dimension 4. For checking incompatibility w.r.t coarse-graining in $\mathbb{C}^4$, the same procedure is repeated taking two POVM measurements with four outcomes each. The corresponding measurements are $\{M'_i\}, \{N'_j\}, i, j \in \{1,2,3,4\}$:

$$\begin{aligned} M'_i &= \nu \left| i \right\rangle \left\langle i \right| + (1-\nu)\frac{\mathbb{1}}{4}, \\ N'_j &= \nu \left| \psi'_j \right\rangle \left\langle \psi'_j \right| + (1-\nu)\frac{\mathbb{1}}{4}, \end{aligned} \tag{3.29}$$

with

$$\begin{aligned} \left| \psi'_0 \right\rangle &= \frac{1}{2}(\left|0\right\rangle + \left|1\right\rangle + \left|2\right\rangle + \left|3\right\rangle), \\ \left| \psi'_1 \right\rangle &= \frac{1}{2}(\left|0\right\rangle + \left|1\right\rangle - \left|2\right\rangle - \left|3\right\rangle), \\ \left| \psi'_2 \right\rangle &= \frac{1}{2}(\left|0\right\rangle - \left|1\right\rangle - \left|2\right\rangle + \left|3\right\rangle), \\ \left| \psi'_3 \right\rangle &= \frac{1}{2}(\left|0\right\rangle - \left|1\right\rangle + \left|2\right\rangle - \left|3\right\rangle). \end{aligned} \tag{3.30}$$

These measurements in Eqs.(3.29) and (3.30) are incompatible (or equivalently 4-incompatible, using the definition 2) for $\nu > 0.666$. In this case, any non-trivial coarse-graining will reduce the measurement outcomes to three or two, and hence we shall have 3-incompatiblity and 2-incompatiblity respectively. For $0.666 < \nu \leq 0.675$, the incompatible measurements become compatible after all possible coarse-grainings that reduce the outcomes to at least three. Further reducing the number of outcomes by coarse-graining gives a tighter bound for the critical value of the robustness parameter above which the measurements are incompatible. Hence, for $0.675 < \nu \leq 0.720$, the measurements are 3-incompatible but become compatible if one further reduces the number of outcomes by coarse-graining. These measurements exhibit the strongest form of incompatibility for $\nu > 0.720$ (as also discussed in Observation 5), i.e., they remain incompatible after all possible non-trivial coarse-grainings. The results for coarse-graining of measurement outcomes in dimensions 3 and 4 are summarized in TABLE 3.1.

	In dimension 3	**In dimension** 4
4-**incompatible**	N.A.	0.666
3-**incompatible**	0.683	0.675
2-**incompatible**	0.711	0.720

TABLE 3.1 Critical values of robustness for MUBs w.r.t. coarse-graining in $\mathbb{C}^3$ and $\mathbb{C}^4$, above which the measurements are incompatible.

3.4.2 Robustness of incompatibility under disjoint-convex-mixing of measurements

Dimension 3**.** For checking incompatibility w.r.t. disjoint-convex-mixing (CM), a minimum of three measurements are needed. The two measurements are $\{M_i\}$ and $\{N_j\}$, as given in Eqs.(3.27) and Eq.(3.28). The third measurement is $\{R_k\}, k \in \{1,2,3\}$ where

$$R_k = \nu \left|\phi_k\right\rangle \left\langle\phi_k\right| + (1-\nu)\frac{\mathbb{1}}{3}. \tag{3.31}$$

with

$$\begin{aligned} \left|\phi_0\right\rangle &= \frac{\omega\left|0\right\rangle + \left|1\right\rangle + \left|2\right\rangle}{\sqrt{3}}, \ \left|\phi_1\right\rangle = \frac{\left|0\right\rangle + \omega\left|1\right\rangle + \left|2\right\rangle}{\sqrt{3}}, \\ \left|\phi_2\right\rangle &= \frac{\left|0\right\rangle + \left|1\right\rangle + \omega\left|2\right\rangle}{\sqrt{3}}. \end{aligned} \tag{3.32}$$

These three measurements are incompatible (equivalently 3-incompatible, by the definition 4) for $\nu > 0.537$. From these three incompatible measurements, one can effectively reduce the number of measurements to two, by taking a new measurement as the disjoint-convex-mixing of any two measurements from the initial set of three measurements (this can be done in 3 ways), while keeping the third measurement same. This process of disjoint-convex-mixing may reduce the degree of incompatibility, as discussed before and evidenced for $0.537 < \nu \leq 0.764$. This corresponds to the case where the two measurements become compatible after considering all possible disjoint-convex-mixing across all partitions. For $\nu > 0.764$, the measurements are incompatible, suggesting that this is the critical value above which, the measurements are robust against disjoint-convex-mixing.

Dimension 4**.** Consider the measurements $\{M'_i\},\{N'_j\}$ as defined in Eqs. (3.29) and (3.30) and another measurement $\{R'_k\}$ where $i,j,k \in \{1,2,3,4\}$,

$$R'_k = \nu \left|\phi'_k\right\rangle \left\langle\phi'_k\right| + (1-\nu)\frac{\mathbb{1}}{4} \tag{3.33}$$

with

$$
\begin{aligned}
|\phi'_0\rangle &= \frac{1}{2}(|0\rangle - |1\rangle - \mathbb{i}\,|2\rangle - \mathbb{i}\,|3\rangle), \\
|\phi'_1\rangle &= \frac{1}{2}(|0\rangle - |1\rangle + \mathbb{i}\,|2\rangle + \mathbb{i}\,|3\rangle), \\
|\phi'_2\rangle &= \frac{1}{2}(|0\rangle + |1\rangle + \mathbb{i}\,|2\rangle - \mathbb{i}\,|3\rangle), \\
|\phi'_3\rangle &= \frac{1}{2}(|0\rangle + |1\rangle - \mathbb{i}\,|2\rangle + \mathbb{i}\,|3\rangle).
\end{aligned} \tag{3.34}
$$

These three measurements in dimension 4 are incompatible for $\nu > 0.692$. As discussed before, disjoint-convex-mixing makes them compatible and they continue to remain so till $\nu = 0.705$. So, for $0.692 < \nu \leq 0.705$, the three measurements are incompatible, but disjoint-convex-mixing of any two of them makes them compatible. For $\nu > 0.705$, the measurements are incompatible and robust against disjoint-convex-mixing across any partition that effectively brings down the number of measurements to two. The results for disjoint-convex-mixing of measurements in dimensions 3 and 4 are summarized in TABLE 3.2.

	In dimension 3	**In dimension** 4
3-**incompatible**	0.537	0.692
2-**incompatible**	0.764	0.705

TABLE 3.2 Critical values of robustness for MUBs w.r.t. disjoint-convex-mixing in $\mathbb{C}^3$ and $\mathbb{C}^4$, above which the measurements are incompatible.

3.5 Operational witnesses of incompatibility

In this section, we explore how different levels of incompatibility can be witnessed through the input-output statistics derived from characterized devices.

3.5.1 Coarse-graining of outcomes

As we mentioned earlier, measurements that are fully incompatible w.r.t. coarse-graining show stronger incompatibility compared to the measurements that are not fully incompatible w.r.t. coarse-graining. To operationally certify whether a set of measurements is fully incompatible w.r.t. coarse-graining is important from the perspective of determining the practical utility of such a set for revealing phenomena such as Bell inequality, steering and contextuality [52–58, 62, 63, 121], as well as for checking the proficiency of such a set for information processing tasks [42, 24]. Certification means that we need to infer the incompatibility of the measurements from the input-output statistics of the measurement device without knowing its internal functioning. Below we study the two classes of certification separately.

Device-independent witness

Violation of Bell inequality provides device-independent witness for incompatible measurements. Here, one neither requires any prior knowledge of the internal functioning of the measurement device nor any idea of the dimension of the system on which the measurements act. However, not all incompatible measurements yield violations of Bell inequalities [74, 75]. Nonetheless, we have the following results.

Result 8. *Full-incompatibility w.r.t. coarse-graining of any two measurements can always be witnessed in a device-independent way.*

Proof. In [53], it has been proven that any pair of binary-outcome incompatible measurements violate at least one Bell-CHSH inequality by suitably choosing a shared entangled state between the parties and suitably choosing measurements on the other subsystem. On the other hand, from Observation 5, we know if two measurements are full-incompatible w.r.t. coarse-graining, then they must be two-incompatible w.r.t. coarse-graining. Combining these two facts, we can conclude this observation. □

Device-independent witness from a single Bell experiment

In the above-mentioned approach, different Bell experiments are probed for different coarse-graining. It is interesting to explore whether different levels of incompatibility w.r.t. coarse-graining can be witnessed universally through a single Bell experiment in a device-independent manner. In this case, the objective is to consider different coarse-grained statistics of a Bell experiment and check whether those statistics have local explanations or not.

To understand how this technique works, we consider a well-known example of two three-outcome rank-one projective measurements in $\mathbb{C}^3$ which give maximum violation of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [122], a well-studied [123–125] Bell-type inequality. The projective measurements of Alice are: $A_a \equiv \{|\xi\rangle_{A,a}\}$, where

$$|\xi\rangle_{A,a} = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \exp\left(\mathbb{i}\frac{2\pi}{3} j(\xi + \alpha_a)\right) |j\rangle_A, \tag{3.35}$$

with $a \in \{1,2\}$ corresponding to two different measurement settings of Alice, and $\xi \in \{0,1,2\}, \alpha_1 = 0, \alpha_2 = \frac{1}{2}$. The projective measurements of Bob are: $B_b \equiv \{|\eta\rangle_{B,b}\}$, where

$$|\eta\rangle_{B,b} = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \exp\left(\mathbb{i}\frac{2\pi}{3} j(-\eta + \beta_b)\right) |j\rangle_B, \tag{3.36}$$

with, $b \in \{1,2\}$ corresponding to two different measurement settings of Bob, and $\eta \in \{0,1,2\}, \beta_1 = \frac{1}{4}, \beta_2 = -\frac{1}{4}$.

Specifically, here we investigate the different layers of incompatibility of the measurements given in (3.35) under coarse-graining. The incompatibility of the measurements of Eq.(3.35) before coarse-graining is guaranteed by the CGLMP-inequality violation. Now we give a theoretical analysis to investigate the incompatibility status of the measurements of Alice given in (3.35) after all possible coarse-grainings. The results are summarized in TABLE 3.3.

Theoretical explanation.— Out of three outcomes if any of the two outcomes are coarse-grained it becomes a (2,2,2,3) scenario, i.e., two inputs for Alice and two inputs for Bob and each input of Alice has two outcomes and each input of Bob has three outcomes. For this scenario, Collins and Gisin have shown that there are a total of 72 CH-facet inequalities [126]. The CH-inequalities [127] are of the form:

$$-1 \leq S \leq 1, \tag{3.37}$$

with CH-functional

$$\begin{aligned} S = \quad & P(00|A_1,B_1) + P(00|A_1,B_2) + P(00|A_2,B_2) \\ & -P(00|A_2,B_1) - P(0|A_1) - P(0|B_2). \end{aligned} \tag{3.38}$$

The other CH-inequalities are obtained by (1) interchanging A_1 with A_2, (2) interchanging B_1 with B_2, and (3) interchanging both A_1 with A_2 and B_1 with B_2. Consider the scenario where we coarse-grain (0,1) outcomes for both the inputs of Alice. Lets make the following relabeling $(0,1) \equiv \overline{0}$ and $2 \equiv \overline{1}$ for the outcomes of A_1 and A_2 and also consider the clubbing of $(0,1) \equiv \overline{0}$ outcomes both for B_1 and B_2. Under this relabelling Eq.(3.37) takes the form:

$$\begin{aligned} -1 \leq \quad & P(\overline{0}\,\overline{0}|A_1,B_1) + P(\overline{0}\,\overline{0}|A_1,B_2) + P(\overline{0}\,\overline{0}|A_2,B_2) \\ & -P(\overline{0}\,\overline{0}|A_2,B_1) - P(\overline{0}|A_1) - P(\overline{0}|B_2) \leq 0, \end{aligned} \tag{3.39}$$

where,

$$P(\bar{0}\,\bar{0}|A_i,B_j) = P(00|A_i,B_j) + P(01|A_i,B_j) \\ +P(10|A_i,B_j) + P(11|A_i,B_j), \tag{3.40}$$

$$P(\bar{0}|A_i) = P(0|A_i) + P(1|A_i), \\ P(\bar{0}|B_j) = P(0|B_j) + P(1|B_j),\ i,j \in \{1,2\}. \tag{3.41}$$

Similarly, there are eight other possible clubbings for Bob's measurement outcomes and for each clubbing, we have facet inequalities similar to Eq.(3.39).

One can check that when there are the same coarse-graining of outcomes for both measurement inputs of Alice, we get a CH-inequality violation. Thus, under these coarse-grainings, the two measurements of Alice remain incompatible. When there are different coarse-grainings, for some cases we get CH-violation, but for other cases, we do not get CH-violation. CH-violation under a particular coarse-graining signifies that the measurements are incompatible. However, when there is no CH-violation, we can not conclude anything regarding the incompatibility status. This is depicted in TABLE 3.3.

Experimental realization.— Recently, an experimental study on CGLMP inequality has been performed [108] where the two parties share orbital angular momentum entanglement in a scenario of multiple settings and outcomes. In their experiment, for $d = 3$ they prepared an orbital angular momentum entangled state which is in $\mathbb{C}^3 \otimes \mathbb{C}^3$ of the form:

$$|\psi\rangle = 0.596\,|+1\rangle_A\,|-1\rangle_B + 0.529\,|+2\rangle_A\,|-2\rangle_B \\ +0.604\,|-1\rangle_A\,|+1\rangle_B, \tag{3.42}$$

with $|+l\rangle_A$ corresponds to the orbital angular momentum eigenstate of the signal photon (A) with orbital angular momentum $+l\hbar$ and $|-l\rangle_B$ corresponds the orbital angular momentum eigenstate of the corresponding idler photon (B) with orbital angular momentum $-l\hbar$. The von Neuman measurements of Alice and Bob are given by $A_i \equiv \{|\Gamma_s^i\rangle_A\,\langle\Gamma_s^i|\}$ and $B_j \equiv \{|\Theta_t^j\rangle_B\,\langle\Theta_t^j|\}$ respectively, where

$$|\Gamma_s^i\rangle_A = \frac{1}{\sqrt{3}}(|+1\rangle_A + \omega^{s+\sigma_i}\,|+2\rangle_A + \omega^{2(s+\sigma_i)}\,|-1\rangle_A), \tag{3.43}$$

Possible CG choices of A_1	Possible CG choices of A_2	ΔS_T	ΔS_E	Incompatible
(0,1)	(0,1)	0.126	0.122	yes
(0,1)	(1,2)	0	0	?
(0,1)	(0,2)	0.126	0.122	yes
(1,2)	(0,1)	0.126	0.122	yes
(1,2)	(1,2)	0.126	0.122	yes
(1,2)	(0,2)	0	0	?
(0,2)	(0,1)	0	0	?
(0,2)	(1,2)	0.126	0.122	yes
(0,2)	(0,2)	0.126	0.122	yes

TABLE 3.3 Incompatibility status for all possible coarse-graining (CG) choices of outcomes of Alice's measurements (A_1 and A_2) in the CGLMP scenario. Here ΔS_T refers to the amount of CH-inequality violation obtained in the theory and ΔS_E refers to the CH-inequality violation calculated from the states and measurements *viz.* (3.42) and (3.43) realised in the experiment [108]. Here "yes" denotes that under the particular CG of A_1 and A_2, the measurements can be witnessed to be incompatible as they give CH-inequality violation. On the other hand, "?" denotes that we can not make any conclusion about their incompatibility under those particular CG as there is no violation of CH-inequality in those cases.

and

$$|\Theta_t^j\rangle_B = \frac{1}{\sqrt{3}}(|-1\rangle_B + \omega^{-t-\gamma_j}|-2\rangle_B + \omega^{2(-t-\gamma_j)}|+1\rangle_B), \tag{3.44}$$

with $i,j \in \{1,2\}$ and $s,t \in \{0,1,2\}, \sigma_1 = \frac{1}{4}, \sigma_2 = \frac{3}{4}, \gamma_1 = \frac{1}{2}$ and $\gamma_2 = 0$, $\omega = \exp(\mathbb{1}\frac{2\pi}{3})$. For $d = 3$ the incompatibility status of the measurements (see (3.43)) used in their experimental arrangements before and after coarse-graining is depicted in TABLE 3.3.

Semi-device-independent witness

In the semi-device-independent approach in prepare-and-measure experiments, we do not have any prior knowledge of the internal functioning of the measurement device; however, we assume the dimension of the system on which the measurements act. To witness different levels of incompatibility, here we focus on a class of communication tasks, namely, $(2,\overline{d},d)-$ random access code tasks (RAC). In this task, the sender, Alice, gets two-dit string input message (x_1,x_2)

with $x_1, x_2 \in [\overline{d}]$, and can communicate a $d-$dimensional system to the receiver, Bob, who wants to guess the value of any of the two dits, i.e. x_1 or x_2, randomly. The notation of $(2,\overline{d},d)-$RAC is adopted from [24]. If any two POVMs, each with $\overline{d}$ outcomes acting on $\mathbb{C}^d$ are jointly measurable, the average success probability of this task,

$$P(2,\overline{d},d) \leq P_{CB}(2,\overline{d},d) = \frac{1}{2}\left(1 + \frac{d}{\overline{d}^2}\right), \tag{3.45}$$

where $P_{CB}(2,\overline{d},d)$ is an upper bound on average success probability using two compatible measurements [42, 24]. It turns out any two incompatible rank-one projective measurements can be witnessed through $(2,d,d)-$RAC [61]. We prove the following result for full incompatibility of two 3-outcome rank-one projective measurements w.r.t. coarse-graining.

Theorem 4. *Two 3-outcome rank-one projective measurements, $M = \{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle\}$ and $N = \{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$, can be witnessed to be fully incompatible w.r.t. coarse-graining via RAC if and only if* $0 < |\langle\phi_i|\psi_j\rangle| < \frac{4}{5}, \quad \forall\, i,j = 0,1,2.$

Proof. Without loss of generality, any pair of three outcome rank-one projective measurements can be written up to unitary freedom as,

$$\begin{aligned} M &= \{|0\rangle\langle 0|, |1\rangle\langle 1|, |2\rangle\langle 2|\} \equiv \{|\phi_i\rangle\langle\phi_i|\}, \\ N &= \{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}, \end{aligned} \tag{3.46}$$

where $|\psi_j\rangle = \alpha_j|0\rangle + \beta_j|1\rangle + \gamma_j|2\rangle$.

Let us first consider the coarse-graining of the second and third outcomes for both measurements. So, the new measurement pair becomes:

$$\begin{aligned} M^{(2,3)} &= \{|0\rangle\langle 0|, |1\rangle\langle 1| + |2\rangle\langle 2|\}, \\ N^{(2,3)} &= \{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\}. \end{aligned} \tag{3.47}$$

This pair of measurements appears in $(2,2,3)-$RAC and the compatibility bound in this case is,

$$P_{CB}(2,2,3) = \frac{1}{2}\left(1 + \frac{3}{2^2}\right) = \frac{7}{8} \tag{3.48}$$

by Eq.(3.45) and this bound is tight. $M^{(2,3)}$ and $N^{(2,3)}$ will give advantage in $(2,2,3)-$RAC if

$$\sum_{x,y=0}^{1} ||M_x^{(2,3)} + N_y^{(2,3)}|| = \sum_{i=1}^{4} ||A_i|| > 7, \tag{3.49}$$

with

$$\begin{aligned} A_1 &= M_0^{(2,3)} + N_0^{(2,3)} = \ket{0}\bra{0} + \ket{\psi_1}\bra{\psi_1}, \\ A_2 &= M_0^{(2,3)} + N_1^{(2,3)} = \mathbb{1} + \ket{0}\bra{0} - \ket{\psi_1}\bra{\psi_1}, \\ A_3 &= M_1^{(2,3)} + N_0^{(2,3)} = \mathbb{1} - \ket{0}\bra{0} + \ket{\psi_1}\bra{\psi_1}, \\ A_4 &= M_1^{(2,3)} + N_1^{(2,3)} = 2\mathbb{1} - \ket{0}\bra{0} - \ket{\psi_1}\bra{\psi_1}, \end{aligned} \tag{3.50}$$

and $||.||$ denotes the maximum eigenvalue of an operator. Now, $\ket{\psi_1}$ can be written as follows

$$\begin{gathered} \ket{\psi_1} = \braket{0|\psi_1}\ket{0} + \braket{u|\psi_1}\ket{u} \text{ with} \\ \braket{0|u} = 0 \text{ and } |\braket{0|\psi_1}|^2 + |\braket{u|\psi_1}|^2 = 1, \end{gathered} \tag{3.51}$$

for some vector $\ket{u}$. So, A_1 can be written as a (2×2) matrix in the basis $\{\ket{0}, \ket{u}\}$,

$$A_1 = \begin{pmatrix} 1 + |\braket{0|\psi_1}|^2 & \braket{0|\psi_1}\braket{\psi_1|u} \\ \braket{u|\psi_1}\braket{\psi_1|0} & |\braket{u|\psi_1}|^2 \end{pmatrix}, \tag{3.52}$$

the maximum eigenvalue of A_1 is $1 + |\braket{0|\psi_1}|$. Thus,

$$||A_1|| = 1 + |\braket{0|\psi_1}|. \tag{3.53}$$

Similarly, A_2 can be expressed in a block diagonal matrix in the ortho-normal basis $\{\ket{0}, \ket{u}, \ket{v}\}$,

$$A_2 = \begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix} \text{ where } \Gamma = \begin{pmatrix} 2 + |\braket{0|\psi_1}|^2 & \braket{0|\psi_1}\braket{\psi_1|u} \\ \braket{u|\psi_1}\braket{\psi_1|0} & 1 + |\braket{u|\psi_1}|^2 \end{pmatrix}, \tag{3.54}$$

and

$$||A_2|| = 2 + |\braket{0|\psi_1}|. \tag{3.55}$$

A_3 and A_4 can also be expressed in a block diagonal matrix in the basis $\{\ket{0}, \ket{u}, \ket{v}\}$,

$$A_3 = \begin{pmatrix} \Sigma & 0 \\ 0 & 1 \end{pmatrix}, \Sigma = \begin{pmatrix} |\braket{0|\psi_1}|^2 & \braket{0|\psi_1}\braket{\psi_1|u} \\ \braket{u|\psi_1}\braket{\psi_1|0} & 1 + |\braket{u|\psi_1}|^2 \end{pmatrix}, \tag{3.56}$$

and

$$||A_3|| = 1 + \sqrt{1 - |\langle 0|\psi_1\rangle|^2}. \tag{3.57}$$

$$A_4 = \begin{pmatrix} \Xi & 0 \\ 0 & 2 \end{pmatrix}, \; \Xi = \begin{pmatrix} 1 - |\langle 0|\psi_1\rangle|^2 & -\langle 0|\psi_1\rangle\langle\psi_1|u\rangle \\ -\langle u|\psi_1\rangle\langle\psi_1|0\rangle & 2 - |\langle u|\psi_1\rangle|^2 \end{pmatrix}, \tag{3.58}$$

and $||A_4|| = 2$.

Substituting the values of $||A_1||, \ldots, ||A_4||$ in Eq.(3.49) and after simplification we get, $|\langle 0|\psi_1\rangle| < \frac{4}{5}$. This condition is obtained for a particular coarse-graining. Therefore, to be fully incompatible w.r.t. coarse-graining, $|\langle\phi_i|\psi_j\rangle| < \frac{4}{5} \; \forall \, i,j$, should hold. □

3.5.2 Disjoint-convex-mixing of measurements

We now study the operational witness of incompatibility w.r.t. disjoint-convex-mixing of measurements.

Device-independent witness

Given a set of n measurements, we can make k partitions of it, where $k \in \{2, \cdots, n\}$. If the k number of measurements obtained by the disjoint-convex-mixing of the measurements provides a violation of Bell inequalities for all possible disjoint-convex-mixing and permutations, then the measurements are witnessed to be $k-$incompatible w.r.t. disjoint-convex-mixing in a device-independent way. Using the result that any two binary-outcome incompatible measurements violate the Bell-CHSH inequality [53], we can witness 2-incompatibility w.r.t. disjoint-convex-mixing from any set of binary-outcome measurements.

Semi-device-independent witness

For three or more numbers of measurements, we can witness whether the measurements are fully-incompatible w.r.t. disjoint-convex-mixture in a semi-device independent manner by constructing suitable random access code tasks.

Theorem 5. *Three noisy Pauli measurements of Eq.* (3.19) *with equal noise ($\nu = \nu_0 = \nu_1 = \nu_2$) are witnessed to be fully incompatible w.r.t. disjoint-convex-mixing via RAC if and only if $\sqrt{2/3} < \nu \leqslant 1$.*

Proof. Consider the measurements $\{Q^i\}$ with $i \in \{1,2\}$ for two possible permutations of outcomes, to be formed by the disjoint-convex-mixing of $\{M\}$ and $\{N\}$

as

$$\begin{aligned}Q^1 &= \{p\,M_0 + (1-p)\,N_0,\ pM_1 + (1-p)N_1\},\\ Q^2 &= \{p\,M_0 + (1-p)N_1, p\,M_1 + (1-p)N_0\}.\end{aligned} \tag{3.59}$$

To witness the incompatibility status of (Q^i, R), one can construct a $(2,2,2)-$RAC task. Now, when Bob has to guess Alice's 1st bit, he performs the measurement $\{Q^i\}$ defined by (3.59), which is realized by the disjoint-convex-mixing of measurements $\{M\}$ and $\{N\}$. For the second bit, he performs the measurement $\{R\}$. The maximum average success probability $P(2,2,2)$ that can be obtained in the RAC task is:

$$\begin{aligned}P(2,2,2) &= \frac{1}{8}\sum_{j,k=0}^{1} ||Q^i_j + R_k||\\ &= \frac{1}{4}(2+\sqrt{2\nu^2 - 2p\nu^2 + 2p^2\nu^2}).\end{aligned} \tag{3.60}$$

It can be shown that for all possible disjoint-convex-mixing, the maximum average success probability is the same as Eq. (3.60). Now, to be fully incompatible w.r.t. disjoint-convex-mixing $P(2,2,2) > 3/4$, which implies $\nu^2(p^2 - p + 1) > \frac{1}{2}$ (by Eq.(3.60)). Since the minimum value of $(p^2 - p + 1)$ is $\frac{3}{4}$, so for $\nu > \sqrt{2/3}$ the measurements are fully incompatible w.r.t. disjoint-convex-mixing.

In Observation 3, we have found that this kind of three-qubit measurements with noise become fully incompatible w.r.t. disjoint-convex-mixing for $\nu > \sqrt{2/3}$. So, we can conclude that these three measurements are fully incompatible w.r.t. disjoint-convex-mixing if and only if quantum advantage is obtained in RAC. □

Let us consider another example of three rank-one projective measurements acting on $\mathbb{C}^3$:

$$\begin{aligned}X &= \{|0_x\rangle\langle 0_x|, |1_x\rangle\langle 1_x|, |2_x\rangle\langle 2_x|\},\\ Y &= \{|0_y\rangle\langle 0_y|, |1_y\rangle\langle 1_y|, |2_y\rangle\langle 2_y|\},\\ Z &= \{|0_z\rangle\langle 0_z|, |1_z\rangle\langle 1_z|, |2_z\rangle\langle 2_z|\},\end{aligned} \tag{3.61}$$

where,

$$
\begin{aligned}
|0_x\rangle &= \frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle, \\
|1_x\rangle &= -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle, \\
|2_x\rangle &= \frac{1}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle, \\
|0_y\rangle &= -\frac{\mathbb{i}}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{\mathbb{i}}{2}|2\rangle, \\
|1_y\rangle &= \frac{\mathbb{i}}{\sqrt{2}}|0\rangle + \frac{\mathbb{i}}{\sqrt{2}}|2\rangle, \\
|2_y\rangle &= -\frac{\mathbb{i}}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{\mathbb{i}}{2}|2\rangle, \\
|0_z\rangle &\equiv |0\rangle, |1_z\rangle \equiv |1\rangle, |2_z\rangle \equiv |2\rangle.
\end{aligned}
\tag{3.62}
$$

We take the following disjoint-convex-mixing of the measurements X and Y: $A = \{p|0_x\rangle\langle 0_x| + (1-p)|0_y\rangle\langle 0_y|, p|1_x\rangle\langle 1_x| + (1-p)|1_y\rangle\langle 1_y|, p|2_x\rangle\langle 2_x| + (1-p)|2_y\rangle\langle 2_y|\}$ with $0 \leq p \leq 1$. We now consider the (2,3,3) RAC game involving the two measurements A and Z. The maximum average success probability $P(2,3,3)$ for this RAC task is given by,

$$
P(2,3,3) = \frac{1}{18}\sum_{i,j=0}^{2} ||A_i + Z_j||, \tag{3.63}
$$

which turns out to be greater than $\frac{2}{3}$ i.e. $P_{CB}(2,3,3)$ for all $p \in [0,1]$. Hence, the measurements A and Z are incompatible.

Next, consider the following measurement (taking an arbitrary disjoint-convex-mixing of X and Y) $A_{i,j,k,l,m,n} = \{p|i_x\rangle\langle i_x| + (1-p)|j_y\rangle\langle j_y|, p|k_x\rangle\langle k_x| + (1-p)|l_y\rangle\langle l_y|, p|m_x\rangle\langle m_x| + (1-p)|n_y\rangle\langle n_y|\}$ with $0 \leq p \leq 1$, $i,j,k,l,m,n \in \{0,1,2\}$, $i \neq k$, $k \neq m$, $m \neq i$ and $j \neq l$, $l \neq n$, $n \neq j$. Note that the aforementioned measurement A is denoted by $A_{0,0,1,1,2,2}$ following the present notation. Following a similar calculation, it can be shown that any such $A_{i,j,k,l,m,n}$ is incompatible with Z for all $p \in [0,1]$.

Similarly, taking an arbitrary convex combination of X and Z (or, Y and Z), it can be shown that the new measurement is incompatible with Y (or, X) for all $p \in [0,1]$.

A generic description for the semi-device independent witness of incompatibility.— Consider a measurement assemblage of n measurements having $\overline{d}$ outcomes, $\mathcal{M} = \{M_{z|x}\}_{z,x}$ where $x \in [n], z \in [\overline{d}]$, and the measurements act on $\mathbb{C}^d$. We make k partitions: $S = \{S_i\}_{i=0}^{k-1}$, $\bigcup_i S_i = \mathcal{M}$, and $S_i \bigcap S_j = \phi$, $\forall\, i,j$ with $i \neq j$. Our purpose is to operationally witness the incompatibility of k measurements produced from the disjoint-convex-mixing of n measurements from each of the k partitions. We can construct a $(k,\overline{d},d)-$RAC where Alice has an k dit input message, *viz.*, $x = (x_0, x_1, \ldots, x_{k-1})$. Depending upon the message, she encodes it in a qudit and sends it to Bob, who on the other hand, gets input $y \in [k]$ and accordingly, he has to predict the value of the corresponding bit x_y. He performs the measurement, which is obtained by the disjoint-convex-mixing of the measurements from the partition S_y and declares the outcome of the measurement. Now, if the success probability $P(k,\overline{d},d)$ is greater than $P_{CB}(k,\overline{d},d)$ [24] for all possible disjoint-convex-mixing and permutations of measurement outcomes, we can operationally witness $k-$incompatibility under disjoint-convex-mixing.

3.6 Conclusions

Measurement incompatibility is a feature in quantum theory that a set of measurements cannot be performed jointly on arbitrary systems [29]. It is one of the fundamental ingredients for non-classical correlations and merits of quantum information science. Incompatibility offers a complex structure with different layers as the number of measurements and the dimension on which the measurements act increases [97, 100, 101]. Thus, understanding the different levels of incompatibility with respect to elementary classical operations is of paramount importance from both the foundational and practical perspectives. In this work, we have considered the two most general classical operations *viz.*, coarse-graining of different outcomes of measurements, and disjoint-convex-mixing of different measurements.

Through our present analysis, we have investigated the different levels of incompatibility arising under the above classical operations. Since environmental effects are ubiquitous in practical scenarios, the tolerance thresholds for maintaining measurement incompatibility against noise under classical operations are investigated here. Furthermore, we have developed a method to operationally witness different levels of measurement incompatibility in the device-independent framework involving Bell-type experiments, and also in the semi-device-independent framework involving prepare-and-measure experiments.

Several examples have been provided to illustrate the efficacy of the operational witnesses proposed here. Our results are particularly useful for the purpose of comparing measurements in terms of their degree of incompatibility. For example, measurements that remain fully incompatible with respect to coarse-graining of outcomes (or disjoint-convex-mixing of measurements) show stronger incompatibility compared to those that are not fully incompatible, thus facilitating the legitimate choice of measurements in an information-processing task where a high degree of incompatibility is required. Our work allows for experimental implementation and one can infer more about these layers of incompatibility by employing these operations in the statistics of previously conducted experiments [108]. Further, our formalism can be applied to predict the threshold values of system parameters enabling observations of incompatibility layers in future experiments.

Our present study motivates future work in several open directions. The condition for full incompatibility for two projective measurements with respect to the coarse-graining of outcomes can be extended for more projective measurements. Similarly, the criterion for full incompatibility of projective measurements with respect to disjoint-convex-mixing could be generalized for any set of measurements. It will be interesting to look for examples where the full incompatibility with respect to coarse-graining can be inferred in a device-independent way from single experimental statistics.

Chapter 4

Resource theoretic efficacy of the single copy of a two-qubit entangled state in a sequential network

4.1 Introduction

One of the most counterintuitive features of quantum mechanics is quantum entanglement [128, 18, 129], which leads to nonclassical phenomena like Bell nonlocality [130, 46] and Einstein-Podolsky-Rosen steering [49, 50]. Apart from the foundational significance of probing the incompatibility between quantum mechanical predictions and the local realist descriptions of nature, quantum entanglement serves as a resource for various information processing and communication tasks. To name a few, some such well-established tasks include quantum teleportation [14], quantum dense coding [15], quantum key distribution [131], certification of genuine randomness [132], and quantum random access codes [133].

In a real laboratory set-up, preparation of any quantum resource always faces different types of complications [134] and such difficulties are quantified by the "preparation cost" associated with the preparation dynamics [135]. Moreover, it is an extremely difficult task to prepare quantum resources having a high degree of isolation from environmental interactions [136, 137]. In fact, quantum correlations in independent environments have been shown to decay asymptotically or even to disappear at a finite time under the action of noise [138–140]. Therefore, the efficient use of resources is one of the primary challenges in the backdrop of current endeavour of building quantum technology. To this end, the possibility of recycling the same resource several times is of great advantage. A particular network scenario suitable for this purpose comprises a single copy of a bipartite entangled state with multiple pairs of independent sequential observers in the two spatially separated wings, where each of these observers makes unsharp measurement and delivers the accessed particle to the next observer [141].

Specifically, in [141], the authors showed that at most two independent observers at one wing can violate the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality [142] with an observer on the other wing by sharing a pair of entangled spin-$1/2$ particles. Note that all except the last one in a sequence of multiple observers cannot perform sharp or projective measurements, since it is desired that some amount of entanglement must survive in the post-measured state in order to be utilized by the subsequent observer. It was shown in [143], that the upper bound of two observers on one wing who can share nonlocality of a two-qubit state in the scenario of unbiased measurement settings, is based on the optimality of the unsharp measurement framework [144, 145] with respect to the trade-off between information gain and disturbance in a quantum measurement [146, 147]. It is thus important to employ such measurements in order to obtain optimal performance in the above sequential network scenario.

The issue of sequential detection of different quantum correlations by multiple observers has been investigated both theoretically and experimentally in several subsequent works using the unsharp measurement formalism. Such works include, for example, steering a single system multiple times [148–154], exploring Bell-type nonlocality in various settings [155–164], witnessing entanglement of bipartite and tripartite states [165–167], sharing of nonlocal advantage of quantum coherence [168], and quantum contextuality [169]. Applications of sequential detection of quantum correlations in different information processing tasks have also been reported, e.g., in the context of randomness certification [170], dimension witness [171], quantum random access codes [172–174], quantum teleportation [175], remote state preparation [176], distinguishing quantum predictions from classical simulations with finite memory [177]. Recently, the technique of choosing different sharpness parameters for the different measurement settings of each observer has been proposed for obtaining the possibility of unbounded number of observers sharing Bell-nonlocality [164, 178, 179], and this type of result has also been probed towards random access code generation [180].

The above works have stimulated wide interest in recycling various types of quantum correlations for their use in multi-observer networks. A natural question emerges in this context as to if any quantitative advantage can be gained from the resource theoretic perspective by the reuse of correlations in a single copy of a quantum state. In the present work we answer this question in the affirmative by computing the consumed resource (entanglement in the present context), as well as the resource theoretic cost of measurement [181] required for witnessing entanglement [37, 38]. For this purpose, we consider a single copy of a bipartite two-qubit entangled state that is shared between multiple observers on both the wings, who sequentially access and perform measurements on the state. We first determine the essential figure of merit in this scenario, that is given by the maximum number of observers who can successfully detect entanglement contained in the bipartite two-qubit state. We next compute the entanglement consumed in the detection process, as well as the robustness of the required measurements, and compare these quantities with those obtained by using several pairs of entangled particles shared by the same pairs of observers on both the wings.

Benchmarking of a sequential network from a resource theoretic perspective has remained hitherto unexplored in the literature. By the formalism used here, we are able to quantitatively demonstrate the resource theoretic advantage of a sequential network compared to the case when multiple copies of entangled states are used as resource.

In particular, we consider a scenario involving equal number, say, n numbers of Alices and Bobs (who represent multiple observers here and all Alices are spatially separated from all Bobs) at both wings. First, the pair Alice^1-Bob^1 performs a measurement to detect entanglement of the initially shared state employing a suitable entanglement witness operator [37], and then, Alice^1 passes her particle to Alice^2, whereas, Bob^1 passes his particle to Bob^2. Next, the pair Alice^2-Bob^2 repeats the same procedure of witnessing entanglement, and passes the particles to the next pair in the sequence, i.e., Alice^3-Bob^3, and the process continues. An extension of the above scenario may also be considered for an asymmetric case involving an unequal number of Alices and Bobs. In the latter case the process is similar to the first scenario, except that Alice^1 can be paired not only with Bob^1 but also with Bob^2, Bob^3 and so on. Similarly, other Alices can be paired with multiple Bobs to share and witness entanglement. It is to be noted that in both the scenarios each observer measures independently on her/his subsystem, being ignorant about the measurement choices and outcomes of others. Additionally, we restrict ourselves to protocols without bias[1], where

[1]For a very recent approach on studying recycling of entanglement, see [182].

each observer employs the same value of the sharpness parameter corresponding to all the settings of her/his non-projective measurement.

We next focus towards addressing the main aim of the present work, *viz.*, a resource theoretic comparison of performance of sequential network schemes based on single copy entangled state, with that of schemes based on multi-copy entangled states shared by the same number of observers. In order to compare the sequential and the non-sequential scenarios we utilize the detectability (or visibility) in terms of the expectation value of the entanglement witness operator [38], as well as the information extraction capability of the involved measurement, defined quantitatively via the robustness of measurement [181]. In terms of the above two parameters, we show that the sequential measurement protocol provides advantages in witnessing entanglement by multiple pair of observers in terms of the resources consumed.

In practical situations, several entanglement witnessing based tasks (for example, detecting eavesdropping in quantum key distribution [183], estimating localizable entanglement [184]) may be required to be performed sequentially in order to execute some communication/computation protocol. In the conventional approach involving non-sequential measurements described above, the necessary amount of physical resources increases with increase in number of tasks to be executed. Since, reducing the resource requirements in any quantum device is one of the basic requirements of modern quantum technology, our results can have potential application in realizing commercial quantum devices that can perform multiple tasks with less resource.

We organise the rest of the chapter as follows. In Sec. 4.2, we provide a brief overview of some basic tools employed in our analysis, such as entanglement witness operators, unsharp measurement and robustness of measurements. Next, in Sec. 4.3, we provide details of our symmetric network scenario. The main results regarding the bounds on the number of observers, and the resource theoretic comparisons pertaining to the first scenario are discussed in Sec. 4.4. In Sec. 4.5 we illustrate the resource-theoretic advantage of the sequential scenario through an example of quantum teleportation of three qubits through the use of witness operators for detecting useful entangled states for teleportation. Sec. 4.6 contains an analysis of the asymmetric extension of the above scenario. Finally, in Sec. 4.7 we summarize our results with some concluding discussions.

4.2 Basic tools

In this section, we present the basic ideas of entanglement witness operators, unsharp measurements and robustness of measurements. These concepts will be used later for presenting the main results of this paper.

4.2.1 Entanglement Witness Operators

A Hermitian operator W is called an entanglement witness operator if there exists at least one entangled state $\rho_e \notin \mathcal{S}$ such that $\mathrm{Tr}(W\rho_e) < 0$ and $\mathrm{Tr}(W\rho) \geq 0$ for all $\rho \in \mathcal{S}$ with $\mathcal{S}$ being the set of all separable states [37]. One can find out an entanglement witness operator for each entangled state. However, finding out the optimal entanglement witness operator for a given entangled state is not always easy [38]. For detecting entanglement, one is usually interested in decomposing an entanglement witness operator in terms of local quantum measurements. This enables performing the detection process using local quantum measurements [37].

Consider the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The optimal entanglement witness operator for this state is given by [37],

$$W = \frac{1}{4}\Big(\mathbb{I} \otimes \mathbb{I} + \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\Big). \tag{4.1}$$

The advantage of this entanglement witness operator is that it can be implemented in the laboratory by performing a three correlated local quantum measurements in the bases associated with the Pauli operators $\{\sigma_x, \sigma_y, \sigma_z\}$.

If in the preparation process of the state $|\psi^+\rangle$ some random noise acts, then the resultant state may turn out to be

$$\rho = p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}, \tag{4.2}$$

where $0 < p \leq 1$; $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ denotes white noise and $(1-p)$ is the strength of the noisy process. The entanglement witness operator W remains optimal for the state ρ as well [37].

4.2.2 Optimality of Unsharp Measurements

A fundamental feature of quantum theory is that no information about a system can be obtained without perturbing its state [185]. Projective measurements, also known as strong measurements, are the most informative at the cost of maximally disturbing the initial state. On the other hand, weak measurements are characterised by broad pointer states, and are less informative, but affect lesser the initial state [141], thereby reflecting a nontrivial trade off between information gain and disturbance [146, 147]. In our sequential networks, it is important to employ such weak measurements which optimize the information gain-disturbance trade-off to achieve best performance, rather than some random choice of measurement. In [143], it was shown that unsharp measurement which is an one-parameter Positive Operator-Valued Measure (POVM) satisfies the optimality criteria.

Generalized quantum measurement or POVM [144, 145] is defined by a set of positive operators that add to identity, i.e., $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 \leq E_i \leq \mathbb{I}\}$. Consider the dichotomic observable $\vec{\sigma} \cdot \hat{n}$, which is the spin component observable for qubits along the direction $\hat{n}$. Here $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector composed of three Pauli operators and $\hat{n}$ is a unit vector in $\mathbb{R}^3$. Given the observable $\vec{\sigma} \cdot \hat{n}$, one can define the dichotomic unsharp observable $E^{\lambda}_{\hat{n}} = E^{\lambda}_{+|\hat{n}} - E^{\lambda}_{-|\hat{n}}$ [186] associated with the sharpness parameter $\lambda \in (0,1]$, where

$$E^{\lambda}_{\pm|\hat{n}} = \lambda P_{\pm|\hat{n}} + \frac{1-\lambda}{2}\mathbb{I} \tag{4.3}$$

are the effect operators and satisfy $E^{\lambda}_{+} + E^{\lambda}_{-} = \mathbb{I}, 0 \leq E^{\lambda}_{\pm} \leq \mathbb{I}$.

The probabilities of getting the outcomes $+1$ and -1, when the above unsharp measurement is performed on the state ρ, are given by $\mathrm{Tr}[\rho E^{\lambda}_{+|\hat{n}}]$ and $\mathrm{Tr}[\rho E^{\lambda}_{-|\hat{n}}]$, respectively. The expectation value of $E^{\lambda}_{\hat{n}}$ for a given ρ is defined as,

$$\langle E^{\lambda}_{\hat{n}} \rangle = \mathrm{Tr}[\rho E^{\lambda}_{+|\hat{n}}] - \mathrm{Tr}[\rho E^{\lambda}_{-|\hat{n}}] = \lambda \langle \vec{\sigma} \cdot \hat{n} \rangle, \tag{4.4}$$

where $\langle \vec{\sigma} \cdot \hat{n} \rangle = \mathrm{Tr}[\rho(P_{+|\hat{n}} - P_{-|\hat{n}})]$ denotes the expectation value of the observable $\vec{\sigma} \cdot \hat{n}$ under projective measurement. The post-measurement state can be determined using the generalized von Neumann-Lüders transformation rule [144, 145] as follows,

$$\rho \rightarrow \frac{\sqrt{E^{\lambda}_{\pm|\hat{n}}}\,\rho\,\sqrt{E^{\lambda}_{\pm|\hat{n}}}}{\mathrm{Tr}(\rho E^{\lambda}_{\pm|\hat{n}})}. \tag{4.5}$$

In the weak measurement formalism with broad pointer states, the optimal information gain-disturbance trade-off is characterised by a condition involving two parameters called quality factor (F) and precision (G). An optimal pointer satisfies $F^2 + G^2 = 1$, which implies maximal information gain for a given amount of disturbance [141]. In the unsharp measurement formalism described above, it turns out that $G = \lambda, F = \sqrt{1-\lambda^2}$ [143], thereby satisfying the optimality condition.

4.2.3 Robustness of Measurements

The concept of "Robustness of Measurements" (RoM) has been formulated recently to quantify the informativeness of a measurement [181]. Given a particular measurement E, RoM of E, denoted by $R(E)$, quantifies to what extent E is a resourceful measurement. When a measurement returns an arbitrary outcome i with probability $q(i)$ independent of the quantum state measured, the measurement is called a trivial measurement [181]. Such a measurement has POVM elements E_i with $E_i = q(i)\mathbb{I}$ for all i. It is evident that trivial measurements are not informative or resourceful at all.

RoM is defined as the minimal amount of noise that needs to be added to the measurement such that the measurement becomes a trivial one [181]. Suppose, instead of always performing the measurement $E = \{E_i\}$, one performs a different measurement $F = \{F_i\}$ sometimes. The informativeness of the measurement E can be captured by the minimal probability of this other measurement F that makes the overall measurement trivial. Hence, the RoM can be formally defined as,

$$\begin{aligned} R(E) &= \min_{F,q} r \\ \text{such that} \quad & \frac{E_i + rF_i}{1+r} = q(i)\,\mathbb{I} \quad \forall\, i, \\ F_i \geq 0\, \forall\, i \text{ and } & \sum_i F_i = \mathbb{I}. \end{aligned} \tag{4.6}$$

Here, the minimization is taken over all noise measurements $F = \{F_i\}$ and all probability distributions $q = \{q(i)\}$.

It was shown in [181] that RoM can be written as,

$$R(E) = \sum_i ||E_i||_\infty - 1, \tag{4.7}$$

where $||E_i||_\infty$ is the operator norm of E_i. Note that $||E_i||_\infty$ is equal to the maximum eigen value of $\sqrt{E_i^\dagger E_i}$ [187]. Hence, for the unsharp measurement $E^\lambda_{\hat{n}} \equiv \{E^\lambda_{+|\hat{n}}, E^\lambda_{-|\hat{n}}\}$ defined in Eq.(4.3), we have

$$R(E^\lambda_{\hat{n}}) = \lambda, \tag{4.8}$$

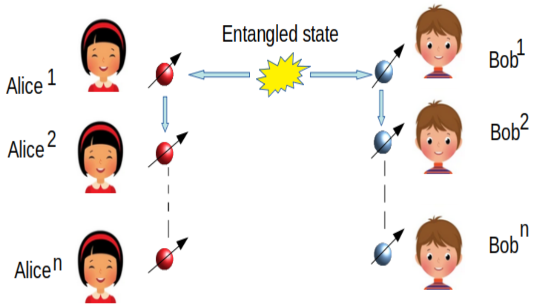

FIG. 4.1 An entangled state ρ_i of two spin-$\frac{1}{2}$ particles is initially shared between Alice^1 and Bob^1, and subsequently, between multiple Alices (Alice^1, Alice^2, Alice^3, $\cdots$) on one wing and multiple Bobs (Bob^1, Bob^2, Bob^3, $\cdots$) on the other wing, sequentially. Multiple Alices perform measurements on one particle sequentially and multiple Bobs perform measurements on the other particle sequentially.

i.e., the sharpness parameter. Using the resource theory of measurement informativeness [181], any trivial measurement can be considered as a free measurement and any measurement which is not trivial is a resourceful measurement. Hence, RoM characterises the amount of resource in a measurement i.e., its information extraction capacity.

4.3 Setting up the scenario

In the present study, we will consider the particular scenario as described below (see Fig. 4.1).

Scenario 1. *We consider n number of sequential Alices (Alice^1, Alice^2, Alice^3, $\cdots$, Alice^n) and n number of sequential Bobs (Bob^1, Bob^2, Bob^3, $\cdots$, Bob^n), where n is a-priori arbitrarily large. At first, the pair Alice^1-Bob^1 detects the entanglement of the initial state ρ_i of two spin-$\frac{1}{2}$ particles using entanglement witness operator. Alice^1 then passes her particle to Alice^2 and Bob^1 passes his particle to Bob^2. Alice^2-Bob^2 then detects entanglement of the two-qubit state received after the measurements by Alice^1 and Bob^1. Consequently, Alice^2 and Bob^2 pass their particles to Alice^3 and Bob^3 respectively, and so on. The process is terminated when the Alice^n-Bob^n pair is unable to detect entanglement.*

In the above scenario we consider the following assumptions:

1) Each Alice (Bob) performs measurements independent of the measurement settings and outcomes of the previous Alices (Bobs).

2) All possible measurement settings of each Alice (Bob) are equally probable, i.e., we are considering unbiased input scenario for each Alice and each Bob.

3) In a given run of the experiment, each Alice (Bob) applies the same value of the sharpness parameter for all of her (his) measurement settings.

In the case of sharp projective measurement, one obtains the maximum amount of information at the cost of maximum disturbance to the state. In the scenarios considered by us, Alice^i (Bob^i) passes on the respective particle to Alice^{i+1} (Bob^{i+1}) after performing suitable measurement. Hence, in this case, Alice^i (Bob^i) needs to perform measurement for detecting entanglement by disturbing the state minimally such that some entanglement remains in the post-measurement state to be detected by Alice^{i+1} (Bob^{i+1}). This can be achieved in the unsharp measurement formalism as the disturbance is minimized for any fixed amount of information gain in this formalism for qubits [143, 148].

4.3.1 Modified entanglement witness operator in unsharp measurement formalism

Note that the entanglement witness operator (4.1) can be implemented in the laboratory by performing projective quantum measurements. Since, in our scenarios, all Alices (Bobs), except the last Alice (Bob) in a sequence, perform unsharp measurements, the entanglement witness operator (4.1) needs to be modified accordingly. In order to modify the entanglement witness operator (4.1), we follow the process described in [165, 166].

Suppose, Alice^i and Bob^j perform unsharp measurement of spin component observables $\vec{\sigma} \cdot \hat{n}_i$ and $\vec{\sigma} \cdot \hat{m}_j$ respectively. The sharpness parameters associated with the measurements by Alice^i and Bob^j are denoted by ξ_i and λ_j respectively with $\xi_i, \lambda_j \in (0,1]$. We will follow this notation throughout the paper.

The joint probability of obtaining the outcomes a_i, b_j (with $a_i, b_j \in \{+1, -1\}$), when Alice^i and Bob^j perform the above unsharp measurements, can be evaluated using the expression,

$$\text{Tr}\Big[\rho\Big(E^{\xi_i}_{a_i|\hat{n}_i} \otimes E^{\lambda_j}_{b_j|\hat{m}_j}\Big)\Big],$$

where ρ is state shared by Alice^i and Bob^j; and the expressions of $E^{\xi_i}_{a_i|\hat{n}_i}$ and $E^{\lambda_j}_{b_j|\hat{m}_j}$ are defined following Eq.(4.3). The expectation value of the the above joint measurement in the state ρ is given by,

$$\begin{aligned}
&\langle E_{\hat{n}_i}^{\xi_i} \otimes E_{\hat{m}_j}^{\lambda_i} \rangle \\
&= \mathbf{Tr}\Big[\Big\{ \Big(E_{+|\hat{n}_i}^{\xi_i} - E_{-|\hat{n}_i}^{\xi_i}\Big) \otimes \Big(E_{+|\hat{m}_j}^{\lambda_j} - E_{-|\hat{m}_j}^{\lambda_j}\Big) \Big\}\rho\Big] \\
&= \xi_i \lambda_j \mathbf{Tr}\Big[\Big\{ \Big(P_{+|\hat{n}_i} - P_{-|\hat{n}_i}\Big) \otimes \Big(P_{+|\hat{m}_j} - P_{-|\hat{m}_j}\Big) \Big\}\rho\Big] \\
&= \xi_i \lambda_j \langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle, \qquad (4.9)
\end{aligned}$$

where $\langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle$ is the expectation value under projective measurements. We can hence use the substitution $\langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle \rightarrow \xi_i \lambda_j \langle \vec{\sigma} \cdot \hat{n}_i \otimes \vec{\sigma} \cdot \hat{m}_j \rangle$ in order to obtain the modified entanglement witness operator for the case of unsharp measurements. For any $\xi_i, \lambda_j \in (0,1]$, the modified entanglement witness operator for the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ has the following form,

$$\begin{aligned}
&W^{(\xi_i,\lambda_j)} \\
&= \frac{1}{4}\Big(\mathbb{I} \otimes \mathbb{I} + \xi_i\sigma_z \otimes \lambda_j\sigma_z - \xi_i\sigma_x \otimes \lambda_j\sigma_x - \xi_i\sigma_y \otimes \lambda_j\sigma_y\Big) \\
&= \frac{1}{4}\Big[\mathbb{I} \otimes \mathbb{I} + \xi_i\lambda_j\Big(\sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\Big)\Big]. \qquad (4.10)
\end{aligned}$$

Now, for any separable state $\rho_s \in \mathcal{S}$, we have

$$\begin{aligned}
\mathbf{Tr}\Big(W^{(\xi_i,\lambda_j)}\rho_s\Big) &= \mathbf{Tr}\Big[\big(\xi_i \lambda_j W + \frac{1}{4}(1 - \xi_i \lambda_j)\mathbb{I} \otimes \mathbb{I}\big)\rho_s\Big] \\
&= \xi_i \lambda_j \mathbf{Tr}\Big(W\rho_s\Big) + \frac{1}{4}(1 - \xi_i \lambda_j) \\
&\geq 0 \quad \forall \rho_s \in \mathcal{S} \ \text{ as } 0 < \xi_i, \lambda_j \leq 1, \qquad (4.11)
\end{aligned}$$

showing that $W^{(\xi_i,\lambda_j)}$ is a valid entanglement witness operator.

4.4 Witnessing entanglement by sequential observers

We now focus on the task of entanglement detection in our sequential network scenario. We take three different types of the initially shared states ρ_i.

4.4.1 Initially shared maximally entangled two-qubit state

Let us consider that Alice^1 and Bob^1 initially share the Bell state given by, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Different pairs of Alice and Bob (i.e., Alice^1-Bob^1, Alice^2-Bob^2, Alice^3-Bob^3, $\cdots$, Alice^n-Bob^n) try to detect entanglement sequentially. At first, we will address the following question: how many such pairs of Alice and Bob can sequentially detect entanglement.

Let Alice^i and Bob^i ($i \in \{1,2,\cdots,n\}$) perform unsharp measurements with sharpness parameters ξ_i and λ_i respectively. The pair Alice^1-Bob^1 can detect entanglement if the following condition is satisfied:

$$\text{Tr}\Big[W^{(\xi_1,\lambda_1)}|\psi^+\rangle\langle\psi^+|\Big] < 0, \tag{4.12}$$

where the operator $W^{(\xi_1,\lambda_1)}$ is given by Eq.(4.10). After simplification, we get the following condition from (4.12),

$$\xi_1\,\lambda_1 > \frac{1}{3} \tag{4.13}$$

Next, let us find out the post measurement state received by the pair Alice^2-Bob^2 from Alice^1-Bob^1. As Alice^2 (Bob^2) acts independent of the measurement setting and outcome of Alice^1 (Bob^1) in each experimental run, we take average over the measurement settings and outcomes by Alice^1 and Bob^1. Hence, the state received, on average, by Alice^2-Bob^2 from Alice^1-Bob^1 is given by,

$$\rho_{A_2B_2} = \frac{1}{9}\sum_{n_1,m_1,a_1,b_1}\Big(\sqrt{E^{\xi_1}_{a_1|\hat{n}_1}}\otimes\sqrt{E^{\lambda_1}_{b_1|\hat{m}_1}}\Big)\,|\psi^+\rangle\langle\psi^+|\,\Big(\sqrt{E^{\xi_1}_{a_1|\hat{n}_1}}\otimes\sqrt{E^{\lambda_1}_{b_1|\hat{m}_1}}\Big), \tag{4.14}$$

with $\hat{n}_1,\hat{m}_1 \in \{\hat{x},\hat{y},\hat{z}\}$ and $a_1,b_1 \in \{+1,-1\}$. Here, we have used the fact that each of Alice^1 and Bob^1 performs any of the three local unsharp measurements associated with the observables σ_x, σ_y, σ_z in each experimental run in order to implement the entanglement witness operator (4.10). We have also used here the assumption that all possible measurement settings of Alice^1 and that of Bob^1 are equally probable. After simplification, we get from Eq.(4.14),

$$\begin{aligned}\rho_{A_2B_2} &= p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2}\\ \text{with } p &= \frac{1}{9}\Big(1+2\sqrt{1-\xi_1^2}\Big)\Big(1+2\sqrt{1-\lambda_1^2}\Big).\end{aligned} \tag{4.15}$$

Since, the state (4.15) has the form given by Eq.(4.31), the Alice^2-Bob^2 pair again uses the same entanglement witness operator given by Eq.(4.10) to detect entanglement. Hence, Alice^2-Bob^2 can detect entanglement if the following condition is satisfied,

$$\mathrm{Tr}\left[W^{(\xi_2,\lambda_2)}\rho_{A_2B_2}\right] < 0, \tag{4.16}$$

which implies the condition,

$$\xi_2\lambda_2 > \frac{3}{\left(1+2\sqrt{1-\xi_1^2}\right)\left(1+2\sqrt{1-\lambda_1^2}\right)}. \tag{4.17}$$

Proceeding in a similar way, it can be shown that the state $\rho_{A_3B_3}$ received, on average, by Alice^3-Bob^3 from Alice^2-Bob^2 has the similar form of Werner state (4.31) and the pair Alice^3-Bob^3 can detect entanglement if

$$\mathrm{Tr}\left[W^{(\xi_3,\lambda_3)}\rho_{A_3B_3}\right] < 0, \tag{4.18}$$

i.e., when

$$\xi_3\lambda_3 > \frac{27}{\Pi_{i=1}^2\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right)}. \tag{4.19}$$

Repeating the above steps, it can be shown that Alice^4-Bob^4 can detect entanglement if the following condition is satisfied,

$$\xi_4\lambda_4 > \frac{243}{\Pi_{i=1}^3\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right)}. \tag{4.20}$$

Similar conditions can be found out that ensure entanglement detection by the other pairs, i.e., Alice^n-Bob^n.

Now, our purpose is to investigate the maximum number of sequential pairs that succeed in witnessing the entanglement of the shared two-qubit state. Combining Eqs.(4.13), (4.17), (4.19) and (4.20) and performing some analytical calculations (see Appendix E for details), we get that Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3 can detect entanglement if the following conditions are satisfied simultaneously,

$$\xi_1 = \lambda_1 = 0.58 + \delta_1 \text{ with } 0 \leq \delta_1 << 1, \tag{4.21}$$

$$\xi_2 = \lambda_2 = 0.66 + \delta_2 \text{ with } 0 \leq \delta_2 << 1, \tag{4.22}$$

$$\xi_3 = \lambda_3 = 0.79 + \delta_3 \text{ with } 0 \leq \delta_3 << 1. \tag{4.23}$$

Here, the numerical digits appearing in the above conditions are rounded to two decimal places.

If Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3 perform measurements with sharpness parameters satisfying Eqs.(4.21) (4.22), (4.23) , then it can be shown that Alice^4-Bob^4 cannot witness the entanglement even if they perform projective measurements, i.e., with $\xi_4 = \lambda_4 = 1$ (see Appendix E for details). Thus, at most three sequential pairs of Alice and Bob can witness the entanglement in this case.

4.4.2 Initially shared two-qubit Werner state

Let Alice^1 and Bob^1 initially share the two-qubit Werner state given by,

$$\rho_i = p|\psi^+\rangle\langle\psi^+| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}, \tag{4.24}$$

where $0 < p \leq 1$. For this state, the optimal entanglement witness operator remains the same as before, i.e., it is W given by Eq.(4.1) [37].

In this case at most three sequential pairs of Alice and Bob (for example, Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3) can witness the entanglement using the entanglement witness operator given by Eq.(4.10). Furthermore, we get the following results,

(1) When $0.80 < p \leq 1$, each of the pairs Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{4, 5, 6, \cdots\}$ cannot detect entanglement.

(2) When $0.57 < p \leq 0.80$, each of the two pairs Alice^1-Bob^1 and Alice^2-Bob^2 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{3, 4, 5, \cdots\}$ cannot detect entanglement.

(3) When $0.33 < p \leq 0.57$, only the pair Alice^1-Bob^1 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{2, 3, 4, \cdots\}$ cannot detect entanglement.

4.4.3 Initially shared non-maximally entangled two-qubit pure state

Suppose that Alice^1 and Bob^1 initially share a non-maximally entangled two-qubit pure state given by,

$$|\Psi\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle, \tag{4.25}$$

with $0 < \theta < \frac{\pi}{4}$. For this state also, the optimal entanglement witness operator is given by Eq.(4.1) [37].

Resource theoretic efficacy of the single copy of a two-qubit entangled state in a sequential network

In this case too, at most three pairs of Alice and Bob (e.g., Alice^1-Bob^1, Alice^2-Bob^2, Alice^3-Bob^3) can witness entanglement sequentially through the entanglement witness operator (4.10). Further, the following results are obtained.

(1) When $\frac{\pi}{8} \leq \theta < \frac{\pi}{4}$, each of the pairs Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{4,5,6,\cdots\}$ cannot detect entanglement.

(2) When $\frac{\pi}{17} \leq \theta < \frac{\pi}{8}$, each of the two pairs Alice^1-Bob^1 and Alice^2-Bob^2 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{3,4,5,\cdots\}$ cannot detect entanglement.

(3) When $0 < \theta < \frac{\pi}{17}$, only the pair Alice^1-Bob^1 can detect the entanglement. Other pairs Alice^i-Bob^i with $i \in \{2,3,4,\cdots\}$ cannot detect entanglement.

4.4.4 Advantage of the sequential measurement scenario

For sequential detection of entanglement by multiple pairs of observers, it is important to ensure that the expectation values of the witness operator for different pairs become as much negative as possible, for feasibility of practical detection of entanglement. In our case, as we discussed earlier, maximum three pairs of Alice and Bob can detect entanglement. Hence, for our purpose, we define 'Detectability' (D) as the minus one times the sum of the expectation values of the entanglement witness operators for all the three pairs.

Mathematically, detectability is defined as

$$D = (-1)\sum_{i=1}^{3} D_{ii} = (-1)\sum_{i} \text{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right]$$
$$\text{with} \quad \text{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right] < 0\,\forall i, \tag{4.26}$$

where $\rho_{A_iB_i}$ is the state shared by the pair Alice^i-Bob^i. Since, negative expectation value of the witness operator implies detection of entanglement, we have taken minus sign in the above definition to make D positive when each pair detects entanglement.

Now it is of practical demand to look for a measurement strategy that would yield optimum witness of entanglement in the sequential measurement scenario. For that we need to define the maximum detectability, D_{max} which is obtained by maximizing D over all possible sharpness parameters (ξ_i, λ_i) of all the three pairs of observers under the constraint that each of the three pairs can detect entanglement, i.e., $\text{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right] < 0\,\forall i \in \{1,2,3\}$.

Mathematically, maximum detectability $D_{\max}$ is defined as,

$$
\begin{aligned}
& D_{\max} = \max_{\xi_1,\lambda_1,\xi_2,\lambda_2,\xi_3,\lambda_3} D \\
& \text{such that} \quad D_{ii} = \mathbf{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right] < 0 \,\forall i \in \{1,2,3\}. \qquad (4.27)
\end{aligned}
$$

$D_{\max}$ serves as a tool to quantify the overall ability of the three pairs of observers with their best measurement strategy to detect entanglement in an experiment in a sequential measurement scenario. Here, the best measurement strategy implies the one that makes the expectation values of the witness operators as much negative as possible for all the three pairs simultaneously. Note that when D in the above definition is maximized, individual $\mathbf{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right]$ may not be optimized. This is because the expectation values of the witness operators of all the three pairs are not optimized simultaneously. Also, for example, when the expectation value of the witness operator for the first pair of Alice and Bob is optimized, other subsequent pairs may not detect any entanglement. Since our objective in the present paper is to optimize the expectation values of the witness operators for all the three pairs simultaneously, we have taken the above definition of maximum detectability.

Also note here that although the above definition is expressed for three pairs of Alice and Bob (relevant for the present study), it can be generalized to any number of pairs of observers depending on the specific context under consideration.

We now demonstrate the advantage of the sequential scenario when Alice[1] and Bob[1] initially share the Bell state $\rho_{A_1B_1} = |\psi^+\rangle\langle\psi^+|$ with $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. In this case, at most three pairs can detect entanglement. Hence, D in this case is given by,

$$
\begin{aligned}
D &= -D_{11} - D_{22} - D_{33} \\
&= (-1)\sum_{i=1}^{3}\mathbf{Tr}\left[W^{(\xi_i,\lambda_i)}\rho_{A_iB_i}\right], \qquad (4.28)
\end{aligned}
$$

where $\rho_{A_iB_i}$ is the state shared, on average, by the pair Alice^i-Bob^i. Based on the analysis described in Sec. 4.4.1, it can be shown that

$$\begin{aligned} D &= -\frac{1}{4}(1-3\xi_1\lambda_1)-\frac{1}{12}\Big[3-\Big(1+2\sqrt{1-\xi_1^2}\Big)\Big(1+2\sqrt{1-\lambda_1^2}\Big)\xi_2\lambda_2\Big] \\ &- \frac{1}{108}\Big[27-\Pi_{i=1}^{2}\Big(1+2\sqrt{1-\xi_i^2}\Big)\Big(1+2\sqrt{1-\lambda_i^2}\Big)\xi_3\lambda_3\Big]. \end{aligned} \tag{4.29}$$

Now, we get $D_{\max}=0.20$, which is obtained for $\xi_1=\lambda_1=0.73$, $\xi_2=\lambda_2=0.80$, and $\xi_3=\lambda_3=1$.

Next, let us evaluate the total RoM that is needed to achieve the above-mentioned $D_{\max}$. As mentioned earlier, for the unsharp measurement $E^{\lambda}_{\hat{n}} \equiv \{E^{\lambda}_{+|\hat{n}}, E^{\lambda}_{-|\hat{n}}\}$ defined by Eq.(4.3), we have $R(M)=\lambda$. Hence, the total RoM, denoted by $R_{\text{total}}(M)$, needed to achieve the above-mentioned $D_{\max}$ is given by,

$$\begin{aligned} R_{\text{total}}(M) &= \sum_{i=1}^{3}(\xi_i+\lambda_i) \text{ such that } D=D_{\max}=0.20 \\ &= 2(0.73+0.80+1) \\ &= 5.06 \end{aligned} \tag{4.30}$$

$R_{\text{total}}(M)$ quantifies the total amount of resource consumed while performing the unsharp measurements necessary for achieving the maximum detectability.

Next, we will compare the resource requirement in terms of the entanglement consumed and total RoM between the sequential measurement scenario (1) and the corresponding non-sequential measurement scenario. The non-sequential measurement scenario involves three pairs of observers- Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3, where each of these three pairs share one copy of a two-qubit entangled state. Hence, this non-sequential measurement scenario involves total three pairs of entangled qubits. Each pair detects entanglement of their shared state using the witness operator (4.10). For the purpose of meaningful comparison of the sequential and the non-sequential scenarios, we need to ensure that the detectability D remains the same for both the scenarios. This will confirm that the overall ability of all the three pairs of observers to detect entanglement is the same in both the scenarios. Next, we will perform the aforementioned comparison when different types of entangled states are shared between the three pairs of Alice and Bob in the non-sequential scenario.

Comparison with non-sequential scenario involving three pairs of entangled qubits in Werner states

Let us first consider that the following Werner state,

$$\rho_{\mathrm{W}}(p_i) = p_i|\psi^+\rangle\langle\psi^+| + (1-p_i)\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2} \tag{4.31}$$

with $0 < p_i \leq 1$ is shared between the pair Alice^i-Bob^i in the aforementioned non-sequential scenario with $i \in \{1,2,3\}$. Here, p_1 p_2 p_3 may or may not be equal to each other. Let $\tilde{\xi}_i$ and $\tilde{\lambda}_i$ denote the sharpness parameters for the measurements by Alice^i and Bob^i respectively in the non-sequential scenario. On the other hand, as mentioned earlier, ξ_i and λ_i are the sharpness parameters for the measurements by Alice^i and Bob^i respectively in the sequential measurement case. Here $\tilde{\xi}_i$ may or may not be equal to ξ_i and $\tilde{\lambda}_i$ may or may not be equal to λ_i for all $i \in \{1,2,3\}$ in general.

Now, we would like to evaluate the minimum amount of the total entanglement (denoted by η) that is necessary in the non-sequential measurement scenario for the detectability in the non-sequential measurement scenario given by, $D^{\mathrm{NS}} = (-1)\sum_{i=1}^{3}\mathrm{Tr}\Big[W^{(\tilde{\xi}_i,\tilde{\lambda}_i)}\rho_{\mathrm{W}}(p_i)\Big] = (-1)\sum_{i=1}^{3}\frac{1}{4}(1-3p_i\tilde{\xi}_i\tilde{\lambda}_i)$ being equal to the maximum detectability in the sequential measurement scenario denoted by $D^{\mathrm{S}}_{\max} = 0.2$ and the total RoM in the non-sequential measurement case given by $\sum_{i=1}^{3}(\tilde{\xi}_i + \tilde{\lambda}_i)$ being equal to that in the sequential measurement case, i.e., equal to 5.06. Also, we must ensure that each of the pairs of Alice and Bob can detect entanglement in the non-sequential scenario while performing the above minimization problem. This last constraint is a natural demand for a meaningful comparison.

The total concurrence of the three copies of the Werner states (4.31) is, $\eta = \sum_{i=1}^{3} C(\rho_{\mathrm{W}}(p_i)) = \sum_{i=1}^{3}\frac{1}{2}(3p_i - 1)$. Thus, now the task is to minimize η with the following constraints: $\sum_{i=1}^{3}(\tilde{\xi}_i + \tilde{\lambda}_i) = 5.06$, $\sum_{i=1}^{3}\frac{1}{4}(1-3p_i\tilde{\xi}_i\tilde{\lambda}_i) = -0.2$ and $\frac{1}{4}(1-3p_i\tilde{\xi}_i\tilde{\lambda}_i) < 0\ \forall\ i \in \{1,2,3\}$. Now it turns out that the minimum of η under the above constraints is given by, $\eta_{\min} = 1.11$. This is achieved when $p_1 = 0.54$, $p_2 = 0.54$, $p_3 = 0.65$, $\tilde{\xi}_1 = \tilde{\lambda}_1 = 0.79$, $\tilde{\xi}_2 = \tilde{\lambda}_2 = 0.79$, $\tilde{\xi}_3 = \tilde{\lambda}_3 = 0.95$. Now, as mentioned earlier, $D^{\mathrm{S}}_{\max}$ in the sequential scenario is achieved for $\xi_1 = \lambda_1 = 0.73$, $\xi_2 = \lambda_2 = 0.80$, $\xi_3 = \lambda_3 = 1$. Hence, we don't need to take equal sharpness parameter in both the scenarios for each of the observers in order to satisfy equal amount of the total RoM in both scenarios and $D^{\mathrm{NS}} = D^{\mathrm{S}}_{\max}$.

		Sequential measurement scenario	Non-sequential measurement scenario
Detectability D		0.20	0.20
Total robustness of measurement $R_{\text{total}}(M)$		5.06	5.06
Total entanglement consumed η	for $\rho_W(p_i)$	1 ebit	≥ 1.11 ebits
	for ρ_{p_i}	1 ebit	≥ 1.11 ebits
	for $\lvert\Psi(\theta_i)\rangle$	1 ebit	≥ 1.11 ebits

TABLE 4.1 The total amount of required entanglement η when each party performs equally resourceful measurements in both scenarios. Here, the non-sequential measurement scenario involves either three pairs of entangled qubits in the Werner states $\rho_W(p_i)$, or three pairs of entangled qubits in the mixed states with colored noise ρ_{p_i}, or three pairs of qubits in non-maximally pure entangled states $|\Psi(\theta_i)\rangle$. All numerical values presented in this table are rounded to two decimal places.

On the other hand, only 1 ebit is sufficient in the sequential measurement scenario for achieving $D^{\text{S}}_{\text{max}}$ as only one copy of the maximally entangled state is involved. Hence, for achieving the same amount of detectability in the sequential and the non-sequential measurement scenarios with the total resourcefulness of the measurements being equal in both the scenarios, the non-sequential measurement scenario needs greater amount of entanglement compared to the sequential measurement scenario. This result is summarized in Table 4.1. This demonstrates a resource theoretic advantage of the sequential measurement scenario over the non-sequential measurement scenario.

It is possible to demonstrate further the advantage of the sequential measurement scenario from another perspective as well. Let us now take the total entanglement consumed η in the two scenarios to be equal. The detectability D^{NS} in the non-sequential measurement scenario is also taken to be equal to the maximum detectability $D^{\text{S}}_{\text{max}}$ in the sequential measurement scenario. We also ensure that each of the pairs of Alice and Bob can detect entanglement in the non-sequential scenario. Under these conditions, we compare the total RoM, i.e., $R_{\text{total}}(M)$ in the two scenarios.

Total entanglement consumption in the sequential measurement scenario starting with a maximally entangled initial state is η = 1 ebit. Hence, for the non-sequential measurement scenario we use three different Werner states, $\rho_{\text{W}}(p_i), i = 1,2,3$ with the total amount of entanglement being equal to that in the

		Sequential measurement scenario	Non-sequential measurement scenario
Detectability D		0.20	0.20
Total entaglement consumed η		1 ebit	1 ebit
Total robustness of measurement $R_{\text{total}}(M)$	for $\rho_{\text{W}}(p_i)$	5.06	≥ 5.16
	for ρ_{p_i}	5.06	≥ 5.17
	for $\lvert\Psi(\theta_i)\rangle$	5.06	≥ 5.17

TABLE 4.2 The required total robustness of measurement $R_{\text{total}}(M)$ when the same amount of total entanglement is consumed in the both scenarios. The non-sequential measurement scenario involves either three pairs of entangled qubits in the Werner states $\rho_W(p_i)$, or three pairs of entangled qubits in the mixed states with colored noise ρ_{p_i}, or three pairs of qubits in non-maximally pure entangled states $|\Psi(\theta_i)\rangle$. All numerical values presented in the table are rounded to two decimal places.

sequential measurement scenario, i.e., $\sum_{i=1}^{3} C(\rho_{\text{W}}(p_i)) = 1$, i.e., $p_1 + p_2 + p_3 = 1.67$. The other constraints are $D^{\text{NS}} = (-1)\sum_{i=1}^{3}\text{Tr}\left[W^{(\tilde{\xi}_i,\tilde{\lambda_i})}\rho_{\text{W}}(p_i)\right] = D^{\text{S}}_{\text{max}} = 0.20$ and $\text{Tr}\left[W^{(\tilde{\xi}_i,\tilde{\lambda}_i)}\rho_{\text{W}}(p_i)\right] < 0 \;\forall\; i \in \{1,2,3\}$. Now, the task is to minimize the total RoM, $R_{\text{total}}(M) = \sum_{i=1}^{3}(\tilde{\xi}_i + \tilde{\lambda}_i)$ in the non-sequential scenario with the aforementioned constraints. It turns out that the minimum total RoM in the non-sequential scenario is 5.16.

On the other hand, the total RoM in the sequential measurement scenario is given by 5.06, as mentioned earlier. Hence, for achieving the same amount of detectability using the same amount of total initial entanglement in the sequential and the non-sequential measurement scenarios, the total resourcefulness of the measurements (in terms of total RoM) necessary in the non-sequential measurement scenario turns out to be greater than that in the sequential measurement scenario. Thus a resource theoretic advantage of the sequential measurement scenario over the non-sequential one is clearly implied. These results are depicted in Table 4.2.

Comparison with non-sequential scenario involving three pairs of entangled qubits in mixed states with colored noise

Now, we consider that the pair Alice^i-Bob^i (with $i \in \{1,2,3\}$) in the non-sequential measurement scenario shares the following mixed state,

$$\rho_{p_i} = p_i|\phi^+\rangle\langle\phi^+| + \frac{1-p_i}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|), \tag{4.32}$$

with $0 < p_i \leq 1$.

Before proceeding, we will construct the optimal entanglement witness operator of the above state using the method described in [37, 188, 189]. First, we compute the eigenvector corresponding to the negative eigenvalue of $\rho_{p_i}^{T_B}$, where $\rho_{p_i}^{T_B}$ denotes the partial transposition of ρ_{p_i}. Then the entanglement witness operator is given by the partially transposed projector onto that eigenvector. Following this approach, we obtain the following optimal entanglement witness operator for the state (4.32),

$$\widetilde{W} = \frac{1}{4}\Big(\mathbb{I}\otimes\mathbb{I} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z\Big). \tag{4.33}$$

Now, suppose in the non-sequential measurement scenario Alice^i performs unsharp measurements with sharpness parameter $\tilde{\xi}_i \in (0,1]$ and Bob^i performs unsharp measurements with sharpness parameter $\tilde{\lambda}_i \in (0,1]$. Following the calculations mentioned in Sec. 4.3.1, it can be shown that the modified entanglement witness operator of the state (4.32) for Alice^i and Bob^i is given by,

$$\widetilde{W}^{(\tilde{\xi}_i,\tilde{\lambda}_i)} = \frac{1}{4}\Big[\mathbb{I}\otimes\mathbb{I} - \tilde{\xi}_i\tilde{\lambda}_i\Big(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\Big)\Big]. \tag{4.34}$$

Following the method mentioned in Sec. 4.3.1, it can be easily shown that for any separable state $\rho_s \in \mathcal{S}$, $\text{Tr}\Big(\widetilde{W}^{(\tilde{\xi}_i,\tilde{\lambda}_i)}\rho_s\Big) \geq 0$ for all $\tilde{\xi}_i.\tilde{\lambda}_i \in (0,1]$.

As mentioned earlier, in the sequential measurement scenario, $D^{\text{S}}_{\text{max}}$ is achieved for $\xi_1 = \lambda_1 = 0.73$, $\xi_2 = \lambda_2 = 0.80$, and $\xi_3 = \lambda_3 = 1$, i.e., total RoM=5.06. Now, we would like to evaluate what is the minimum amount of the total entanglement η that is necessary in the non-sequential measurement scenario for the the total RoM in the non-sequential measurement case being equal to that in the sequential measurement case and for the detectability in the non-sequential measurement scenario given by, $D^{\text{NS}} = (-1)\sum_{i=1}^{3}\text{Tr}\Big[\widetilde{W}^{(\tilde{\xi}_i,\tilde{\lambda}_i)}\rho_{p_i}\Big] = (-1)\sum_{i=1}^{3}\frac{1}{4}\Big[1 + \tilde{\xi}_i\tilde{\lambda}_i(1-4p_i)\Big]$ being equal to $D^{\text{S}}_{\text{max}} = 0.20$. Here also we must ensure that each pair can detect entanglement in the non-sequential scenario. The total concurrence of the three states is, $\eta = \sum_{i=1}^{3} C(\rho_{p_i}) = \sum_{i=1}^{3}(2p_i - 1)$. Thus, now the task is to minimize η with the following constraints: $\sum_{i=1}^{3}(\tilde{\xi}_i + \tilde{\lambda}_i) = 5.06$, $\sum_{i=1}^{3} \frac{1}{4}\Big[1 + \tilde{\xi}_i\tilde{\lambda}_i(1-4p_i)\Big] = -0.2$, and $\frac{1}{4}\Big[1 + \tilde{\xi}_i\tilde{\lambda}_i(1-4p_i)\Big] < 0$ for all $i \in \{1,2,3\}$. Now it turns out that $\eta_{\text{min}} = 1.11$. On the other hand, only 1 ebit is sufficient in the sequential measurement scenario for achieving $D^{\text{S}}_{\text{max}}$ as only one copy of the maximally entangled state is involved. So here also, the total entanglement consumption in the non-sequential scenario is greater than the sequential scenario for achieving the same detectability using equally resourceful measurements in the two scenarios. This result is summarized in Table 4.1. This again demonstrates a resource theoretic advantage of the sequential measurement scenario over the non-sequential measurement scenario.

Next, we take the total entanglement consumed η in the non-sequential scenario to be equal to that in the sequential scenario (= 1 ebit) which gives rise the following constraint, $p_1 + p_2 + p_3 = 2$. Also, the detectability D^{NS} in the non-sequential measurement scenario has to be equal to the maximum detectability $D^{\text{S}}_{\text{max}}$ in the sequential measurement scenario, i.e., $\sum_{i=1}^{3} \frac{1}{4}\Big[1 + \tilde{\xi}_i\tilde{\lambda}_i(1-4p_i)\Big] = -0.2$. Finally, we must ensure that each pair in the non-sequential scenario detects entanglement- $\frac{1}{4}\Big[1 + \tilde{\xi}_i\tilde{\lambda}_i(1-4p_i)\Big] < 0$, $\forall\, i \in \{1,2,3\}$. Under these conditions, we compare the total RoM, i.e., $R_{\text{total}}(M)$ in the two scenarios. Here too, we observe that the minimum amount of total RoM required in the non-sequential measurement scenario is given by, $R^{\text{min}}_{\text{total}}(M) = 5.17$. These results are depicted in Table 4.2. This again signifies a resource theoretic advantage of the sequential measurement scenario over the non-sequential one.

Comparison with non-sequential scenario involving three pairs of qubits in non-maximally entangled pure states

Next, we perform a similar comparison between the sequential and the non-sequential measurement scenario, where the pair Alice^i-Bob^i (with $i \in \{1,2,3\}$) in the non-sequential measurement scenario shares the following non-maximally pure entangled state,

$$|\Psi(\theta_i)\rangle = \cos\theta_i|01\rangle + \sin\theta_i|10\rangle, \tag{4.35}$$

with $0 < \theta_i < \frac{\pi}{4}$. In the non-sequential measurement scenario, each pair of Alice and Bob detects entanglement of the above pure state through the entanglement witness operator given by Eq.(4.10).

Similar to the earlier cases, at first, we calculate the total amount of entanglement η that is necessary in the non-sequential measurement scenario involving three different non-maximally pure entangled states $|\Psi(\theta_i)\rangle$ for the detectability in the non-sequential measurement scenario: $D^{\text{NS}} = \sum_{i=1}^{3} \mathbf{Tr}\left[W^{(\tilde{\xi}_i, \tilde{\lambda}_i)} |\Psi(\theta_i)\rangle\langle\Psi(\theta_i)|\right]$ to be equal to the maximum detectability in the sequential measurement scenario and for the total RoM in the non-sequential measurement scenario to be equal to that of the sequential scenario. Also, we ensure that each pair in the non-sequential case detects entanglement. The results in this case can be obtained following the similar steps as adopted in the previous two cases. These results are summarized in Table 4.1. Here too, it is evident that the sequential measurement scenario is advantageous over the non sequential scenario.

Now, let us consider that the total entanglement consumed η in the non-sequential measurement scenario is equal to that in the sequential measurement scenario (= 1 ebit). Also, consider that the detectability in the non-sequential measurement scenario is equal to the maximum detectability in the sequential measurement scenario and each of the three pairs in the non-sequential case detects entanglement. Under these conditions we have again found that the minimum amount of total measurement resource required in the non-sequential scenario is greater than that in the sequential one. Thus, the advantage of the sequential measurement scenario over the non-sequential one is reinforced.

4.5 Quantum teleportation of three qubits

In this section we illustrate the resource theoretic efficacy of the sequential measurement scheme by presenting an example of quantum teleportation of three qubits. There exist a class of Hermitian witness operators [190] that can detect entangled states useful for performing quantum teleportation [14]. We will show below that a single copy of an entangled state can be recycled such that multiple pairs of observers can detect entangled states suitable for quantum teleportation. This thus demonstrates that if a pair detects entanglement using a quantum state, then the residual entanglement after this entanglement detection can again be used to perform quantum teleportation by another pair of observers. However, in the conventional approach employing non-sequential measurement strategies, more than one copies of entangled states are required for sequentially performing the two tasks- entanglement detection and quantum teleportation.

Here, we consider that the following maximally entangled mixed state [191] is initially shared in the sequential measurement scenario,

$$\rho_{\text{MEMS}} = \begin{pmatrix} h(c) & 0 & 0 & \frac{c}{2} \\ 0 & 1-2h(c) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c}{2} & 0 & 0 & h(c) \end{pmatrix},$$
$$\text{with } h(c) = \begin{cases} \dfrac{c}{2}, & \text{if } c \geq \dfrac{2}{3}, \\ \dfrac{1}{3}, & \text{if } c < \dfrac{2}{3}. \end{cases} \tag{4.36}$$

with c being the concurrence of ρ_{MEMS}. The teleportation witness operator for this state is given by [190],

$$W_{\text{tel}} = \frac{1}{4}\Big(\mathbb{I}\otimes\mathbb{I} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z\Big), \tag{4.37}$$

which satisfies the following properties: $\text{Tr}(W_{\text{tel}}\,\sigma) \geq 0$ for all states σ that are not useful for teleportation and $\text{Tr}(W_{\text{tel}}\,\rho) < 0$ for at least one state ρ that is useful for teleportation.

Now using the same prescription described earlier it can be shown that at most three pairs of Alice and Bob in the aforementioned sequential measurement scenario can get negative expectation values of the witness operator (4.37) when a single copy of the state (4.36) is initially shared. Further, three pairs can have negative expectation values of the witness operator (4.37) when ρ_{MEMS} with concurrence $c \geq 0.93$ is recycled.

Next, in the present context, let us compare the sequential scenario and the corresponding non-sequential scenario. Let in the sequential scenario, Alice^1 and Bob^1 initially share the state ρ_{MEMS} with the minimum amount of necessary concurrence such that three pairs can get negative expectation values of the witness operator (4.37), i.e., with concurrence $c = 0.93$. Following the similar calculations as described earlier it can be shown that the maximum detectability in the sequential case is given by, $D^{\text{S}}_{\text{max}} = 0.11$. This occurs for total RoM being equal to 4.96.

Next, in the non-sequential scenario, we consider that each pair Alice^i-Bob^i with $i \in \{1,2,3\}$ shares the state ρ_{MEMS} given by (4.36) with concurrence c_i. Now, we compute the total amount of entanglement $\eta = c_1 + c_2 + c_3$ that is necessary in the non-sequential measurement scenario for the detectability in the non-sequential measurement scenario being equal to the maximum detectability in the sequential measurement scenario and for the total RoM in the non-sequential measurement scenario to be equal to that in the sequential scenario. Also, we ensure that each pair in the non-sequential case detects useful state for teleportation. It turns out that minimum total entanglement consumed in the non-sequential scenario is 1.98 ebits, thus clearly implying an advantage of the sequential scenario.

Therefore, this analysis implies that the sequential scenario requires less resource compared to the corresponding non-sequential scenario for executing sequential detection of entangled states or useful states for quantum teleportation.

4.6 More number of observers in an asymmetric scenario

In the scenario considered so far in this work, we assume that the number of sequential Alices on one side is equal to the number of sequential Bobs on the other. We call this scenario as a "Symmetric Scenario". At this stage it may be pertinent to ask the question as to what happens if we relax this condition of symmetry. To this end, let us now consider sequential detection of entanglement in the case when the number of Alices is not equal in general to the number of Bobs. We consider that the Bell state given by, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is initially shared.

It may be noted here that an asymmetric scenario involving a single Alice and multiple Bobs (Bob^1, Bob^2, Bob^3, $\cdots$, Bob^m) was discussed in [165]. In this case, it was shown that at most twelve Bobs can detect entanglement with the single Alice [165].

Next, consider the scenario involving two Alices (Alice^1 and Alice^2) and multiple Bobs (Bob^1, Bob^2, Bob^3, $\cdots$, Bob^m). At first, the pair Alice^1-Bob^1 detects entanglement using the entanglement witness operator (4.10). As mentioned in Sec. 4.4.1, this pair can detect entanglement if the condition (4.13) is satisfied. The state $\rho_{A_2B_2}$ received, on average, by Alice^2-Bob^2 from Alice^1-Bob^1 is given by Eq.(4.15). Since, there are only two Alices in the sequence, Alice^2 performs projective measurements. On the other hand, all subsequent Bobs (i.e., Bob^2, Bob^3, $\cdots$, Bob^m) perform unsharp measurements. The sharpness parameter associated with the measurement by Bob^i (with $i \in \{2, 3, \cdots, m\}$) is denoted by λ_i. Hence, the modified entanglement witness operator used by the pair Alice^2-Bob^i

is given by,

$$W^{(\lambda_i)} = \frac{1}{4}\Big[\mathbb{I}\otimes\mathbb{I} + \lambda_i\Big(\sigma_z\otimes\sigma_z - \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y\Big)\Big]. \tag{4.38}$$

This can be obtained from the entanglement witness operator given by Eq.(4.1) and following the analysis mentioned in Sec. 4.3.1. It can be shown that for any separable state $\rho_s \in \mathcal{S}$, we have $\text{Tr}\Big(W^{(\lambda_j)}\rho_s\Big) \geq 0$.

Alice^2 and Bob^2 can witness entanglement if

$$\textbf{Tr}\Big[W^{(\lambda_2)}\rho_{A_2B_2}\Big] < 0, \tag{4.39}$$

which implies the following condition,

$$\lambda_2 > \frac{3}{\Big(1+2\sqrt{1-\xi_1^2}\Big)\Big(1+2\sqrt{1-\lambda_1^2}\Big)}. \tag{4.40}$$

The average state shared between Alice^2 and Bob^3 is given by,

$$\rho_{A_2B_3} = \frac{1}{3}\sum_{m_2,b_2}\Big(\mathbb{I}\otimes\sqrt{E^{\lambda_2}_{b_2|\hat{m}_2}}\Big)\rho_{A_2B_2}\Big(\mathbb{I}\otimes\sqrt{E^{\lambda_2}_{b_2|\hat{m}_2}}\Big), \tag{4.41}$$

with $\hat{m}_2 \in \{\hat{x},\hat{y},\hat{z}\}$ and $b_2 \in \{+1,-1\}$. Here also, we use the assumption that all possible measurement settings of Bob^2 are equally probable.

Similarly, Alice^2 and Bob^3 can witness entanglement if the following condition is satisfied,

$$\textbf{Tr}\Big[W^{(\lambda_3)}\rho_{A_2B_3}\Big] < 0. \tag{4.42}$$

The above condition implies the following,

$$\lambda_3 > \frac{9}{\Big(1+2\sqrt{1-\xi_1^2}\Big)\Big(1+2\sqrt{1-\lambda_1^2}\Big)\Big(1+2\sqrt{1-\lambda_2^2}\Big)}. \tag{4.43}$$

Repeating the above calculations for other subsequent Bobs (i.e., Bob^4, Bob^5, $\cdots$, Bob^m), we can derive the conditions on λ_4, λ_5, $\cdots$.

Next, we determine the maximum number of Bobs that succeed in witnessing the entanglement of the shared state with Alice^2. Combining the conditions (4.13), (4.40), (4.43), other similar conditions on λ_4, λ_5, $\cdots$, and performing analytical calculations as described in Appendix E, we get that Bob^2, Bob^3, $\cdots$, Bob^8 can detect entanglement with Alice^2, if the following conditions are satisfied simultaneously,

$$
\begin{aligned}
\xi_1 &= \lambda_1 = 0.58 + \tilde{\delta_1} \text{ with } 0 \leq \tilde{\delta_1} << 1, \\
\lambda_2 &= 0.44 + \tilde{\delta_2} \text{ with } 0 \leq \tilde{\delta_2} << 1, \\
\lambda_3 &= 0.47 + \tilde{\delta_3} \text{ with } 0 \leq \tilde{\delta_3} << 1, \\
\lambda_4 &= 0.51 + \tilde{\delta_4} \text{ with } 0 \leq \tilde{\delta_4} << 1, \\
\lambda_5 &= 0.56 + \tilde{\delta_5} \text{ with } 0 \leq \tilde{\delta_5} << 1, \\
\lambda_6 &= 0.63 + \tilde{\delta_6} \text{ with } 0 \leq \tilde{\delta_6} << 1, \\
\lambda_7 &= 0.74 + \tilde{\delta_7} \text{ with } 0 \leq \tilde{\delta_7} << 1, \\
\lambda_8 &= 0.95 + \tilde{\delta_8} \text{ with } 0 \leq \tilde{\delta_8} << 1.
\end{aligned} \tag{4.44}
$$

Here the numerical values appearing in the above conditions are rounded to two decimal places.

If Alice^1, Alice^2, Bob^1, Bob^2, Bob^3, $\cdots$, Bob^8 perform particular measurements with sharpness parameters satisfying the conditions mentioned in (4.44), then it can be shown that Bob^9 cannot witness entanglement with Alice^2 even if Bob^9 performs projective measurements, i.e., with $\lambda_9 = 1$. Hence, at most eight sequential Bobs can detect entanglement in this scenario with two Alices.

Next, let us consider the scenario involving three Alices (Alice^1, Alice^2 and Alice^3) and multiple Bobs (Bob^1, Bob^2, Bob^3, $\cdots$, Bob^m). At first, the pair Alice^1-Bob^1 detects entanglement using the entanglement witness operator (4.10). Then Alice^1 and Bob^1 pass their particles to Alice^2 and Bob^2, respectively. The pair Alice^2-Bob^2 detects entanglement using the same entanglement witness operator (4.10) and passes the respective particles to the pair Alice^3-Bob^3 who performs measurements to detect entanglement. Now, Bob^3 passes his particle to Bob^4 so that the pair Alice^3-Bob^4 can detect entanglement. Subsequently, Bob^4 passes the particle to Bob^5, and so on. In this scenario, following the aforementioned calculations, it can be shown that Bob^3, Bob^4 and Bob^5 can detect entanglement with Alice^3. No additional Bob can detect entanglement with Alice^3. Hence, at most five Bobs can detect entanglement.

Finally, if we consider that there are four Alices (Alice^1, Alice^2, Alice^3 and Alice^4) and multiple Bobs (Bob^1, Bob^2, Bob^3, $\cdots$, Bob^m), then the result derived in Sec. 4.4.1 for the symmetric scenario tells us that it is not possible for four pairs (i.e., Alice^1-Bob^1, Alice^2-Bob^2, Alice^3-Bob^3 and Alice^4-Bob^4) to detect entanglement sequentially. Entanglement detection is possible only up to the third pair.

Before concluding, it may be noted that a resource theoretic comparison of the above asymmetric scenarios can be performed with the corresponding non-sequential scenarios involving multiple copies of mixed or non-maximally pure entangled initial states. As expected, similar to the case of the symmetric scenario, here too it is possible to observe advantages of the sequential scenario in terms of the entanglement consumed and robustness of measurement.

4.7 Concluding Discussions

To summarize, our analysis presented here clearly shows a hitherto unexplored resource theoretic advantage of recycling a single copy of a two-qubit quantum state towards detecting entanglement by multiple observers. Specifically, we have analyzed in detail a scenario involving multiple independent observers acting sequentially on each of the two spatially separated wings that initially share the resource of a single copy of a two-qubit entangled state. The number of observers on the two wings may be equal or unequal, depending upon the type of network considered. We have estimated the maximum number of observers that can detect entanglement sequentially using only one pair of qubits. Our results indicate that one can reduce the number of physical qubits needed in the context of performing different entanglement-assisted quantum tasks multiple times in network scenarios.

Furthermore, we have performed a quantitative analysis of the advantage of the sequential measurement scenario involving multiple sequential observers on both wings from the perspective of resource requirements. In particular, we have shown that the above scenario can help in reducing the necessary requirement of the total initial entanglement as well as the resource cost of the measurements necessary for detecting entanglement by multiple sequential observers. These advantages demonstrate the benefits of recycling single-shot entanglement in various entanglement-assisted quantum information processing and communication tasks in practical contexts, in comparison with various standard schemes employing either multiple copies of two-qubit entangled states, or multipartite entangled states for performing such tasks.

The analysis of this work may be extended to certain interesting directions. The scheme of unsharp measurement that we have adopted in this work employs the same value of the sharpness parameter associated with measurement in any direction by an individual observer in a given run of the experiment. Considering different sharpness parameters for different measurement input settings by an observer can lead to a significant increase in the number of allowed observers [164, 178, 179]. The resource theoretic efficacy of such a scheme would then be worthwhile to study. A fertile direction of study could also be to generalize the scheme formulated in the present paper towards studying resource theoretic efficacy of sequential sharing of single-shot entanglement in the context of multi-qubit and two-qudit higher dimensional entangled states. Finally, categorizing various information processing tasks involving sequential measurements [170, 177, 171–176, 180] in terms of their resource theoretic advantages is another potentially attractive direction of future study.

Before concluding, it may be noted that several entanglement-witness based tasks may be required to be performed sequentially in practical situations. This can be achieved in one of the following two ways: (i) by using different copies of quantum states for different rounds of tasks, or (ii) by recycling the same copy of a quantum states. Obviously, the second approach requires less number of physical systems. However, from a resource theoretic point of view, it had been unclear before the analysis of our present work whether the second method really consumes less resource (either in terms of total entanglement, or in terms of total resourcefulness of all the measurements performed). This is because it may so happen that the total amount of entanglement of all quantum states necessary in the first approach is smaller than the necessary entanglement of the single copy of the quantum state needed in the second approach. It is this void in the literature that the present paper seeks to fill in by establishing a resource centric advantage of the second method. However, the present study does not capture the advantage of the sequential scenario from all perspectives. For example, sharing one copy of an entangled state and sharing multiple copies of entangled states between spatially separated observers require different experimental efforts. Therefore, probing a more general resource based comparison between sequential and non-sequential scenario incorporating all such factors related to practical implementations is worth for future research and our present analysis indeed motivates future studies along this direction.

Chapter 5

Summary and possible open problems

In summary, the thesis explores two fundamental pillars of quantum information science: quantum measurements, particularly incompatible measurements, and quantum entanglement. We have shown in Chapter 2 that incompatible measurements are indeed a valuable resource, as they are necessary for achieving quantum advantage in any communication task within the prepare-and-measure scenario. As it turns out that incompatible measurements are a potential candidate for various information processing tasks, it is of practical importance to certify them. Here, we construct a method for semi-device-independent certification of the incompatibility of any finite number of measurements with arbitrary but finite outcomes by suitably constructing random access codes (RAC) task.

Then, in Chapter 3 we give a hierarchy of incompatible measurements under elementary classical operations that can be easily performed on the input and output of the measurement device. These will be useful for comparing different sets of incompatible measurements based on their degree of incompatibility, and can set a benchmark to choose incompatible measurements for various information processing tasks legitimately. We also consider the ubiquitous presence of noise in the real-world scenario, which deteriorates the degree of incompatibility, and we evaluate the critical amount of noise that different layers of incompatibility can withstand with respect to these classical operations. We discuss how different layers of incompatibility under these classical operations can be experimentally witnessed both from device-independent and semi-device-independent ways.

Finally, in Chapter 4 we discuss how entanglement can be recycled in a sequential network scenario such that it will be economical in terms of resource consumption compared to the non-sequential counterpart, where entanglement is not recycled, rather multiple copies of the entangled state are used. This technique can be useful for quantum information processing tasks in the quantum network scenario.

There are also several open directions to explore in future:

1. We have given an operational method to witness incompatible measurements. However, our method has a limitation: it can not conclusively witness incompatible measurements every time. Are there other efficient ways to operationally witness incompatible measurements conclusively in the prepare and measure scenario?

2. Our method of operational witness of incompatibility may be generalised to semi-device-independent witness of channel incompatibility and instrument incompatibility.

3. Also, the operational witness of all incompatible extremal POVMs is a possible open problem.

4. We have given a quantifier of measurement incompatibility. Are there any other better measures of incompatibility? Recently, in [100], a quantifier of incompatibility has been presented. Here, they quantify the incompatibility in terms of the dimension of the classical system required to simulate the statistics of incompatible measurements. It would be interesting to compare this quantifier with ours.

5. Recently, it has been shown in [192] that incompatible qubit measurements can always be device-independently certified as they always provide a multipartite Bell inequality violation. What is the status of incompatible measurements in higher dimensions in this regard?

Appendix A

Proof of Result 2

Proof. For a given set of measurements $\{M_{b_y|y}\}$ on Bob's side, the maximum average success probability (2.16) in quantum theory is given by [61]

$$\begin{aligned}
& \max_{\{\rho_x\}} \frac{1}{n\prod_y d_y} \sum_{x_1x_2\cdots x_n} \sum_y \mathbf{Tr}\left(\rho_x M_{b_y=x_y|y}\right) \\
= & \frac{1}{n\prod_y d_y} \sum_{x_1x_2\cdots x_n} \max_{\{\rho_x\}} \mathbf{Tr}\left(\rho_x \left(\sum_y M_{b_y=x_y|y}\right)\right) \\
= & \frac{1}{n\prod_y d_y} \sum_{x_1x_2\cdots x_n} ||\chi||, \qquad \text{(A.1)}
\end{aligned}$$

where,

$$\chi = M_{x_1|1} + M_{x_2|2} + \cdots + M_{x_n|n}. \qquad \text{(A.2)}$$

Here, $||\chi||$ denotes the operator norm of χ, which is simply the maximum eigenvalue of the operator χ. Our aim is to obtain an upper bound on the expression (A.1) when $\{M_{b_y|y}\}$ are compatible.

An alternative definition of measurement incompatibility, which is equivalent to the standard one (1.2), is associated with the existence of a parent POVM whose appropriate marginals give rise to all the individual measurements [29]. Precisely, if the measurements $\{M_{b_y|y}\}$ are compatible then there exists a parent measurement, $G \equiv \{G(b_1, \cdots, b_n)\}$, with $\prod_{y=1}^{n} d_y$ elements from which all the measurement operators can be reconstructed by taking marginals as follows

$$M_{x_y|y} = \sum_{b_1,\cdots,b_{y-1},b_{y+1},\cdots,b_n} G(b_1, \cdots, b_{y_1}, x_y, b_{y+1}, \cdots, b_n), \qquad \text{(A.3)}$$

where

$$\sum_{b_1,\cdots,b_n} G(b_1,\cdots,b_{y_1},b_y,b_{y+1},\cdots,b_n) = \mathbb{1}_{d\times d}. \tag{A.4}$$

Let us first expand χ in terms of the parent POVM using (A.3),

$$\chi = \sum_{b_2,b_3,\cdots,b_n} G(x_1,b_2,b_3\cdots,b_n) + \quad \textstyle\sum_{b_1,b_3,\cdots,b_n} G(b_1,x_2,b_3,\cdots,b_n) + \cdots$$
$$+ \quad \textstyle\sum_{b_1,b_2,b_3,\cdots,b_{n-1}} G(b_1,b_2,b_3,\cdots,b_{n-1},x_n). \tag{A.5}$$

Each term in the above expansion can be split into two terms in the following way

$$\begin{aligned}\chi \;=\; & \sum_{b_3,b_4,\cdots,b_n} G(x_1,x_2,b_3,b_4,\cdots,b_n) + \sum_{\substack{b_2,\cdots,b_n\\ b_2\neq x_2}} G(x_1,b_2,\cdots,b_n)\\ & + \sum_{b_1,b_4,\cdots,b_n} G(b_1,x_2,x_3,b_4,\cdots,b_n) + \sum_{\substack{b_1,b_3,\cdots,b_n\\ b_3\neq x_3}} G(b_1,x_2,b_3,\cdots,b_n)\\ & +\cdots\\ & + \sum_{b_2,b_3,\cdots,b_{n-1}} G(x_1,b_2,b_3,\cdots,b_{n-1},x_n) + \sum_{\substack{b_1,\cdots,b_{n-1}\\ b_1\neq x_1}} G(b_1,\cdots,b_{n-1},x_n).\end{aligned} \tag{A.6}$$

In the above Eq. (A.6), there are two sums in each line, and there is a total of n lines. Let us denote the first sum and the second sum in the i^{th} line by $\mathcal{S}_1^i$ and $\mathcal{S}_2^i$ respectively, where $i \in \{1,\cdots,n\}$. Hence, Eq. (A.6) can be expressed as

$$\chi = \sum_{i=1}^{n} \left(\mathcal{S}_1^i + \mathcal{S}_2^i\right), \tag{A.7}$$

where

$$\mathcal{S}_1^i = \sum_{b_1,\cdots,b_{i-1},b_{i+2},\cdots,b_n} G(b_1,\cdots,b_{i-1},x_i,x_{i+1},b_{i+2},\cdots,b_n), \tag{A.8}$$

and

$$\mathcal{S}_2^i = \sum_{\substack{b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n\\ b_{i+1}\neq x_{i+1}}} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_n). \tag{A.9}$$

Here, the index i is taken to be modulo n. Each $G(\cdots)$ in the above sums will be termed as an element.

Let us now make an observation that there is no common element between the $\mathcal{S}_2^i$ and $\mathcal{S}_2^{i+1}$. The common element between $\mathcal{S}_2^i$ and $\mathcal{S}_2^j$ with $i, j \in \{1, \cdots, n\}$ and $j > i+1$ is

$$\begin{aligned}
&\sum_{\substack{b_1,\cdots,b_n \\ b_{i+1}\neq x_{i+1} \\ b_{j+1}\neq x_{j+1}}} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \\
&\leqslant \sum_{\substack{k=1 \\ k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \qquad \text{(A.10)}
\end{aligned}$$

where the index i, j is taken to be modulo n. Hence, we have

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{S}_2^i \quad &\leqslant \quad \sum_{\substack{i,j\in\{1,\cdots,n\} \\ j>i+1}} \left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \right) + \mathcal{O}, \\
\text{where,} \quad \mathcal{O} = &\quad \text{other terms with no common element.} \qquad \text{(A.11)}
\end{aligned}$$

Next, let us focus on $\mathcal{S}_1^i$. It can be checked that

$$\sum_{i=1}^{n} \mathcal{S}_1^i = \sum_{i=1}^{n} \left(\sum_{\substack{k=1 \\ k\neq i,i+1}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,x_{i+1},b_{i+2},\cdots,b_n) \right). \qquad \text{(A.12)}$$

Replacing $\mathcal{S}_2^i$ and $\mathcal{S}_1^i$ using (A.11) and (A.12) in (A.7), we have

$$\begin{aligned}
\chi \quad &\leqslant \quad \sum_{\substack{i,j\in\{1,\cdots,n\} \\ i<j}} \left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \right) + \mathcal{O} \\
&\leqslant \quad \sum_{\substack{i,j\in\{1,\cdots,n\} \\ i<j}} \left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \right) \\
&+ \quad \sum_{b_1,\cdots,b_n} G(b_1,\cdots,b_n). \qquad \text{(A.13)}
\end{aligned}$$

Substituting the above expression into (A.1) and employing the triangle inequality for the norm, we find that

$$S_c(n,\vec{d},d) \leqslant \frac{1}{n\prod_y d_y}\left(\sum_{x_1,\cdots,x_n}\sum_{\substack{i,j\in\{1,\cdots,n\}\\ i<j}}\Big|\Big|\sum_{\substack{k=1\\ k\neq i,j}}^{n}\sum_{b_k}G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n)\Big|\Big| + \sum_{x_1,\cdots,x_n}\Big|\Big|\sum_{b_1,\cdots,b_n}G(b_1,\cdots,b_n)\Big|\Big|\right). \tag{A.14}$$

Due to (A.4), the second term of the above expression can be evaluated as

$$\sum_{x_1,\cdots,x_n}\Big|\Big|\sum_{b_1,\cdots,b_n}G(b_1,\cdots,b_n)\Big|\Big| = \sum_{x_1,\cdots,x_n}\big|\big|\mathbb{1}_{d\times d}\big|\big| = \prod_{y=1}^{n}d_y. \tag{A.15}$$

Next, consider the first term in (A.14), given by

$$\sum_{x_1,\cdots,x_n}\sum_{\substack{i,j\in\{1,\cdots,n\}\\ i<j}}\Big|\Big|\sum_{\substack{k=1\\ k\neq i,j}}^{n}\sum_{b_k}G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n)\Big|\Big| = \sum_{\substack{i,j\in\{1,\cdots,n\}\\ i<j}}\beta_{i,j} \tag{A.16}$$

where

$$
\begin{aligned}
\beta_{i,j} &= \sum_{x_1,\cdots,x_n} \Big|\Big| \sum_{\substack{k=1\\k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \Big|\Big| \\
&\leqslant \sum_{x_1,\cdots,x_n} \mathbf{Tr} \left(\sum_{\substack{k=1\\k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \right) \\
&= \sum_{\substack{r=1\\r\neq i,j}}^{n} \sum_{x_r} \mathbf{Tr} \left(\sum_{x_i,x_j} \sum_{\substack{k=1\\k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \right) \\
&= \sum_{\substack{r=1\\r\neq i,j}}^{n} \sum_{x_r} \mathbf{Tr} \left(\sum_{b_1,\cdots,b_n} G(b_1,\cdots,b_n) \right) \\
&= \sum_{\substack{r=1\\r\neq i,j}}^{n} \sum_{x_r} \mathbf{Tr}(\mathbb{1}_{d\times d}) \\
&= d \prod_{\substack{y=1\\y\neq i,j}}^{n} d_y. \qquad \text{(A.17)}
\end{aligned}
$$

Hence, we get the first term in (A.14), given by

$$
\sum_{x_1,\cdots,x_n} \sum_{\substack{i,j\in\{1,\cdots,n\}\\i<j}} \Big|\Big| \sum_{\substack{k=1\\k\neq i,j}}^{n} \sum_{b_k} G(b_1,\cdots,b_{i-1},x_i,b_{i+1},\cdots,b_{j-1},x_j,b_{j+1},\cdots,b_n) \Big|\Big| = d \sum_{\substack{i,j\in\{1,\cdots,n\}\\i<j}} \prod_{\substack{y=1\\y\neq i,j}}^{n} d_y \qquad \text{(A.18)}
$$

By substituting the bounds from (A.15)-(A.18) into (A.14), we obtain

$$
S_c(n,\vec{d},d) \leqslant \frac{1}{n\prod_y d_y} \left(d \sum_{\substack{i,j\in\{1,\cdots,n\}\\i<j}} \left(\prod_{\substack{y=1\\y\neq i,j}}^{n} d_y \right) + \prod_{y=1}^{n} d_y \right), \qquad \text{(A.19)}
$$

which reduces to the first expression of Eq. (2.18).

For the other bound, let us use the fact that in (A.5), only the term $G(x_1,x_2,\cdots,x_n)$ occurs n times and all the other terms can occur, at most, $(n-1)$ times to get an upper bound on χ as follows

$$\chi \leqslant G(x_1, x_2, \cdots, x_n) + (n-1) \sum_{b_1, \cdots, b_n} G(b_1, \cdots, b_n). \tag{A.20}$$

Replacing this bound into (A.4) and employing the triangle inequality for the norm, we get

$$S_c(n, \vec{d}, d) \leqslant \frac{1}{n \prod_y d_y} \left(\sum_{x_1, \cdots, x_n} ||G(x_1, \cdots, x_n)|| + (n-1) \sum_{x_1, \cdots, x_n} || \sum_{b_1, \cdots, b_n} G(b_1, \cdots, b_n)|| \right). \tag{A.21}$$

We already have a bound given by (A.15) on the second sum in the above equation. The first term is bounded by d since

$$\begin{aligned} \sum_{x_1, \cdots, x_n} ||G(x_1, \cdots, x_n)|| &\leqslant \sum_{x_1, \cdots, x_n} \mathbf{Tr}\left(G(x_1, \cdots, x_n)\right) \\ &= \mathbf{Tr} \left(\sum_{x_1, \cdots, x_n} G(x_1, \cdots, x_n) \right) \\ &= \mathbf{Tr}\left(\mathbb{1}_{d \times d}\right) \\ &= d. \end{aligned} \tag{A.22}$$

Therefore, we arrive at

$$S_c(n, \vec{d}, d) \leqslant \frac{1}{n \prod_y d_y} \left(d + (n-1) \prod_y d_y \right), \tag{A.23}$$

which reduces to the second expression of Eq. (2.18). This completes the proof.

□

Appendix B

Proof of result 3

In order to provide a detailed proof of Result 3, we first state a general feature of communication tasks.

Lemma 1. *Consider a general form of a linear function of* $\{p(b_y|x,y)\}$,

$$S = \sum_{x,y,b_y} c_{x,y,b_y} p(b_y|x,y). \tag{B.1}$$

The maximum value of S *within* $\mathcal{C}_d$*, which we denote by* S_c*, is obtained by deterministic strategies and can be written only in terms of the decoding function* $\{p_b(b_y|y,m)\}$.

Proof. Replacing the expression of $p(b_y|x,y)$ for classical communication given by Eq. (2.1) into (B.1), we see that

$$\begin{aligned}
S_c &= \max_{\substack{\{p_a(m|x,\lambda)\} \\ \{p_b(b_y|y,m,\lambda)\} \\ \{\pi(\lambda)\}}} \sum_x \left(\sum_{y,b_y} c_{x,y,b_y} p(b_y|x,y) \right) \\
&= \max_{\substack{\{p_a(m|x,\lambda)\} \\ \{p_b(b_y|y,m,\lambda)\} \\ \{\pi(\lambda)\}}} \int_\lambda \pi(\lambda) \left[\sum_x \left\{ \sum_m p_a(m|x,\lambda) \left(\sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m,\lambda) \right) \right\} \right] d\lambda \\
&= \max_{\substack{\{p_b(b_y|y,m,\lambda)\} \\ \{\pi(\lambda)\}}} \int_\lambda \pi(\lambda) \left[\sum_x \max_m \left(\sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m,\lambda) \right) \right] d\lambda.
\end{aligned} \tag{B.2}$$

This is achieved when $p_a(m^*(\lambda,x)|x,\lambda)=1$, where for each λ, $m^*(\lambda,x)$ is defined as follows

$$\sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m^*(\lambda,x),\lambda) \geq \sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m,\lambda)\ \forall\, m\in[d]. \tag{B.3}$$

Now, the above expression (B.2) is a convex sum with respect to $\pi(\lambda)$ and thus, we can omit the dependence of λ by taking the best value of $\left[\sum_x \max_m \left(\sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m,\lambda)\right)\right]$ over different choices of λ as follows:

$$S_c = \max_{\{p_b(b_y|y,m)\}} \left[\sum_x \max_m \left(\sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m)\right)\right]. \tag{B.4}$$

Therefore, it is sufficient to consider deterministic decoding, that is, $p_b(b_y|y,m)\in\{0,1\}$ to achieve S_c. Moreover, given any decoding strategy $\{p_b(b_y|y,m)\}$, the best encoding function is

$$p_a(m^*|x)=1,\ \text{where} \sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m^*) \geq \sum_{y,b_y} c_{x,y,b_y} p_b(b_y|y,m)\ \forall\, m\in[d]. \tag{B.5}$$

This completes the proof. □

Proof of Result 3. The proof is essentially a generalization of the proof given in Section II-A of [88], which was restricted for the particular case where $d=d_y$ for all y. We know from the above *lemma* that the optimal encoding and decoding functions are deterministic. Thus, this can be written in a functional form as

$$E(x_1\cdots x_n)=m\ \text{ if } p_a(m|x)=1, \tag{B.6}$$

and

$$D_y(m)=b_y\ \text{ if }\ p_b(b_y|y,m)=1. \tag{B.7}$$

Here, $E(x_1\cdots x_n)$ is a function whose domain is the set of inputs $x=x_1\cdots x_n$ and range is the set of messages $[d]$. Also, $D_y(m)$ is a function whose domain is the set of messages $[d]$ and range is the set $[d_y]$. We say the decoding strategy is 'identity decoding', denoted by $\{\tilde{D}_y\}$, if

$$\forall y,\ \tilde{D}_y(m)=m. \tag{B.8}$$

We want to show that, without loss of generality we can take $\{\tilde{D}_y\}$ for the maximum success probability. Consider an encoding $E(x)$ (B.6) and a decoding $\{D_y\}$ (B.7) that may not be $\{\tilde{D}_y\}$, that is, there may exists y such that $D_y(m) \neq m$. Let $D_y^{\leftarrow}(b_y)$ be the preimage of b_y, that is, $D_y^{\leftarrow}(b_y) = \{m \in [d] : D_y(m) = b_y\}$.

Subsequently, we consider the following quantity

$$D_1^{\leftarrow}(b_1)\cdots D_n^{\leftarrow}(b_n) = \{m_1\cdots m_n : D_1(m_1) = b_1, \cdots, D_n(m_n) = b_n\}, \tag{B.9}$$

which is simply the set of dit-string $\{m_1\cdots m_n\}$ that is mapped to the sting $b_1\cdots b_n$. We define another encoding function $\{\tilde{E}_x\}$ as follows

$$\tilde{E}(D_1^{\leftarrow}(x_1)D_2^{\leftarrow}(x_2)\cdots D_n^{\leftarrow}(x_n)) = m \text{ if } E(x_1\cdots x_n) = m. \tag{B.10}$$

The above definition of $\tilde{E}$ is not complete since it is not defined if $x_i \notin [d]$ since $D_y^{\leftarrow}(x_i) \in [d]$. In those cases, we take any encoding strategy. Now, we note that $\tilde{E}$ is a well-defined encoding function. Also note that $\tilde{E}$ is a valid encoding for the random access codes considered by us only if $d \leqslant \min_y d_y$. This is because, if $d > d_y$ for some y, then the domain of $\tilde{E}$ may have a string of n dits that does not belong to x.

Suppose, for any input pair $x_1\cdots x_n, y$ so that the encoding E and decoding $\{D_y\}$ guesses the correct dit x_y. Hence, if the encoding strategy is given by, $E(x = x_1\cdots x_n) = m$, then the decoding strategy is given by, $D_y(m) = b_y = x_y$. Therefore, we have $D_y^{\leftarrow}(x_y) = m$. As a consequence, the new encoding $\tilde{E}_x$ and the 'identify decoding' $\{\tilde{D}_y\}$ also provides the correct answer for at least one input pair from $\{D_1^{\leftarrow}(x_1)D_2^{\leftarrow}(x_2)\cdots D_n^{\leftarrow}(x_n)\}, y$. Hence, the average success probability for the strategy consisting of the encoding $\tilde{E}_x$ and the 'identify decoding' $\{\tilde{D}_y\}$ is greater than or equal to that for the strategy with encoding E and decoding $\{D_y\}$. Therefore, we can consider 'identity decoding' without loss of generality.

Next, from Eq.(B.4), the expression for S_c pertaining to the random access codes for 'identity decoding' can be written as

$$S_c = \frac{1}{n\prod_y d_y}\sum_x \max_m \left(\sum_y P(b_y = x_y|y,m)\right) = \frac{1}{n\prod_y d_y}\sum_x \max_m \left(\sum_y \delta_{x_y,m}\right), \tag{B.11}$$

and for the 'identity decoding', the best encoding can be determined from (B.5) as follows

$$p_a(m^*|x) = 1, \text{ where } \sum_y \delta_{x_y,m^*} \geq \sum_y \delta_{x_y,m} \;\forall\, m \in [d]. \tag{B.12}$$

Hence, the best encoding pertaining to the 'identity decoding' is just sending the dit that belongs to $[d]$ and occurs maximum times in the input string $x_1 \cdots x_n$.

Finally, we provide an expression for S_c for the best classical strategy derived above. In an input string $x_1 \cdots x_n$, say, the dit i occurs n_i number of times. The maximum value of a dit can be $\max_y d_y$. Alice sends message m such that $n_m = \max_{i=1,\cdots,d} n_i$. As a result, out of n different values of y, they get success $(\max_{i=1,\cdots,d} n_i)$ times. As the total number of dits is n, the set of values of n_i should satisfy

$$\sum_{i=1}^{d_{\max}} n_i = n, \tag{B.13}$$

where $d_{\max} = \max_y d_y$. Moreover, dit i may not belong to all $[d_y]$ and thus, n_i can not take all the solutions of the above equation. Say, k_i is the number of sets among $[d_1], \cdots, [d_n]$ such that dit $i \in [d_y]$. Therefore, we are only interested in those solutions where $n_i \leqslant k_i$.

Given such a solution of $\{n_i\}$, there will be many possible numbers of input dit strings x having that $\{n_i\}$. Next, let us evaluate the number of input dit strings x that can have an arbitrary $\{n_i\}$. In any input string, at most $k_{d_{\max}}$ number of input dits can have the value $d_{\max}$. In the given set of input dit strings having $\{n_i\}$, the dit $d_{\max}$ occurs $n_{d_{\max}}$ number of times. Hence, the dit $d_{\max}$ can be arranged in $C^{k_{d_{\max}}}_{n_{d_{\max}}}$ different possible ways. Next, in any input string, at most $k_{d_{\max}-1}$ number of input dits can have the value $(d_{\max} - 1)$. However, among these $k_{d_{\max}-1}$ number of input dits, $n_{d_{\max}}$ number of dits have already taken the value $d_{\max}$ in the case of the given set of input strings. Also, in the given set of input dit strings having $\{n_i\}$, the dit $(d_{\max} - 1)$ occurs $n_{d_{\max}-1}$ number of times. Therefore, for any of the above-mentioned arrangements of the dit $d_{\max}$, the dit $(d_{\max} - 1)$ can be arranged in $C^{k_{d_{\max}-1} - n_{d_{\max}}}_{n_{d_{\max}-1}}$ different possible ways. Proceeding in this way, it can be shown that an arbitrary dit j can be arranged in $C^{\alpha_j}_{n_j}$ different possible ways with $\alpha_j = k_j - \sum_{i=j+1}^{d_{\max}} n_i$ for any arrangement of the dits- $d_{\max}$, $(d_{\max} - 1)$, $\cdots$, $j+1$. Therefore, given any $\{n_i\}$, there will be $\left(\prod_{j=1}^{d_{\max}} C^{\alpha_j}_{n_j}\right)$ (with $\alpha_j = k_j - \sum_{i=j+1}^{d_{\max}} n_i$) number of input dit strings having that $\{n_i\}$. Combining these facts we obtain Eq. (2.21). $\square$

Appendix C

Derivation of Eq. (2.24) and Eq. (2.25)

From Result 3, we can write the following for $n = 2$, $d_y = \overline{d}$ for all y, and $d \leqslant \overline{d}$,

$$S_c(2,\overline{d},d) = \frac{1}{2\overline{d}^2} \sum_{\{n_i\}\in\mathcal{S}} \left[N_{\{n_i\}} \max_{i=1,\cdots,d} \{n_i\} \right] \tag{C.1}$$

where $N_{\{n_i\}}$ is the number of input dit strings having a given $\{n_i\}$; and $\mathcal{S}$ denotes the set of $\{n_i\}$ satisfying

$$\sum_{i=1}^{\overline{d}} n_i = 2 \tag{C.2}$$

such that $n_i \leqslant 2$ for all i.

Next, let us characterize the set $\mathcal{S}$. It can be noted that there are the following two types of $\{n_i\} \in \mathcal{S}$:

1. For each $i \in [\overline{d}]$, $n_i = 2$ and $n_j = 0$ for all $j \neq i$ and $j \in [\overline{d}]$.

   There are $\overline{d}$ number of such $\{n_i\} \in \mathcal{S}$. However, $\max_{i=1,\cdots,d}\{n_i\} = 0$ for each of those $\{n_i\} \in \mathcal{S}$ satisfying $n_i = 2$ for any i such that $i \in \{d+1,\cdots,\overline{d}\}$ and $n_j = 0$ for all $j \in [\overline{d}]$ and $j \neq i$. Hence, only d number of $\{n_i\} \in \mathcal{S}$ belonging to this class contribute to the sum of (C.1). It is straightforward to check that for each of these d number of $\{n_i\} \in \mathcal{S}$, $N_{\{n_i\}} = 1$ and $\max_{i=1,\cdots,d}\{n_i\} = 2$.

2. For each $i, j \in [\overline{d}]$ with $i \neq j$, $n_i = n_j = 1$ and $n_k = 0$ for all $k \notin \{i, j\}$ with $k \in [\overline{d}]$.

   There are $C_2^{\overline{d}}$ number of such $\{n_i\} \in \mathcal{S}$. However, $\max_{i=1,\cdots,d}\{n_i\} = 0$ for each of those $\{n_i\} \in \mathcal{S}$ satisfying $n_i = n_j = 1$ for any i, j with $i \neq j$, $i, j \in \{d+1,\cdots,\overline{d}\}$ and $n_k = 0$ for all $k \in [\overline{d}]$, $k \neq i$, $k \neq j$. There are $C_2^{(\overline{d}-d)}$ number of such $\{n_i\} \in \mathcal{S}$ satisfying this. Hence, only $C_2^{\overline{d}} - C_2^{(\overline{d}-d)}$ number of $\{n_i\} \in \mathcal{S}$ belonging to this second class contribute to the sum of (C.1). It can be checked that for each of these $C_2^{\bar{d}} - C_2^{(\bar{d}-d)}$ number of $\{n_i\} \in \mathcal{S}$, $N_{\{n_i\}} = 2$ and $\max_{i=1,\cdots,d}\{n_i\} = 1$.

Therefore, we have from Eq. (C.1)

$$\begin{aligned} S_c(2,\bar{d},d) &= \frac{1}{2\bar{d}^2}\left[2d + 2\left(C_2^{\bar{d}} - C_2^{(\bar{d}-d)}\right)\right] \\ &= \frac{1}{2\bar{d}^2}\left[d + 2d\bar{d} - d^2\right]. \end{aligned} \tag{C.3}$$

Similarly, following the same analysis as above, we can get the expression for $n = 3$, $d_y = \bar{d}$ for all y, and $d \leqslant \bar{d}$,

$$S_c(3,\bar{d},d) = \frac{1}{3\bar{d}^3} \sum_{\{n_i\}\in\mathcal{S}} \left[N_{\{n_i\}} \max_{i=1,\cdots,d}\{n_i\}\right] \tag{C.4}$$

where $N_{\{n_i\}}$ is the number of input dit strings with a given $\{n_i\}$; and $\mathcal{S}$ denotes the set of $\{n_i\}$ satisfying

$$\sum_{i=1}^{\bar{d}} n_i = 3 \tag{C.5}$$

such that $n_i \leqslant 3$ for all i. Now there are three cases that satisfy Eq. (C.5):

1. For each $i \in [\bar{d}]$, $n_i = 3$ and $n_j = 0$ for all $j \neq i$ and $j \in [\bar{d}]$.

   There are $\bar{d}$ number of such $\{n_i\} \in \mathcal{S}$. However, $\max_{i=1,\cdots,d}\{n_i\} = 0$ for each of those $\{n_i\} \in \mathcal{S}$ satisfying $n_i = 3$ for any i with $i \in \{d+1,\cdots,\bar{d}\}$ and $n_j = 0$ for all $j \in [\bar{d}]$ and $j \neq i$. Hence, only d number of $\{n_i\} \in \mathcal{S}$ belonging to this class contribute to the sum of (C.4). For each of these d number of $\{n_i\} \in \mathcal{S}$, we have that $N_{\{n_i\}} = 1$ and $\max_{i=1,\cdots,d}\{n_i\} = 3$. Hence, the contribution to the sum is $3d$.

2. For each $i,j,k \in [\bar{d}]$ with $i \notin \{j,k\}$, $j \notin \{i,k\}$, $k \notin \{i,j\}$, $n_i = n_j = n_k = 1$ and $n_l = 0$ for all $l \notin \{i,j,k\}$ and $l \in [\bar{d}]$.

   There are $C_3^{\bar{d}}$ number of such $\{n_i\} \in \mathcal{S}$. Moreover, $\max_{i=1,\cdots,d}\{n_i\} = 0$ for each of those $\{n_i\} \in \mathcal{S}$ satisfying $n_i = n_j = n_k = 1$ for any choice of i,j,k with $i,j,k \in \{d+1,\cdots,\bar{d}\}$, $i \notin \{j,k\}$, $j \notin \{i,k\}$, $k \notin \{i,j\}$ and $n_l = 0$ for all $l \in [\bar{d}]$ and $l \notin \{i,j,k\}$. There are $C_3^{(\bar{d}-d)}$ number of such $\{n_i\} \in \mathcal{S}$ satisfying this. Thus, only $C_3^{\bar{d}} - C_3^{(\bar{d}-d)}$ number of $\{n_i\} \in \mathcal{S}$ belonging to this class contribute to the sum of (C.4). It can be checked that for each of these $C_3^{\bar{d}} - C_3^{(\bar{d}-d)}$ number of

$\{n_i\} \in \mathcal{S}$, $N_{\{n_i\}} = 3!$ and $\max_{i=1,\cdots,d}\{n_i\} = 1$. Therefore, the contribution to the sum will be $(3!)\left(C_3^{\overline{d}} - C_3^{\overline{d}-d}\right)$.

3. For each $i,j \in [\overline{d}]$ with $i \neq j$, $n_i = 2$, $n_j = 1$ and $n_k = 0$ for all $k \notin \{i,j\}$ with $k \in [\overline{d}]$.

   The feasible solutions of Eq.(C.5) that contribute to Eq.(C.4) are of two types:

   (A) $i \in [d]$ and $j \in [\overline{d}] - \{i\}$. The number of possible such $\{n_i\} \in \mathcal{S}$ is given by, $d(\overline{d}-1)$. Also, for each such $\{n_i\}$, we have that $N_{\{n_i\}} = 3$ and $\max_{i=1,\cdots,d}\{n_i\} = 2$. Therefore, the contribution to the sum appearing in Eq. (C.4) by this case is $6d(\overline{d}-1)$.

   (B) $i \in \{d+1,\cdots,\overline{d}\}$ and $j \in [d]$. The number of possible such $\{n_i\} \in \mathcal{S}$ is given by, $d(\overline{d}-d)$. And for each such $\{n_i\}$, we have that $N_{\{n_i\}} = 3$ and $\max_{i=1,\cdots,d}\{n_i\} = 1$. Hence, the contribution to the sum appearing in Eq. (C.4) for this case is given by, $3d(\overline{d}-d)$.

   Therefore, the total contribution to the sum of Eq. (C.4) is given by, $6d(\overline{d}-1) + 3d(\overline{d}-d) = 3d(3\overline{d}-d-2)$.

Therefore, we have

$$S_c(3,\overline{d},d) = \frac{1}{3\overline{d}^3}\left[(3!)\left(C_3^{\overline{d}} - C_3^{\overline{d}-d}\right) + 3d(3\overline{d}-d-2) + 3d\right]$$
$$= \frac{d}{3\overline{d}^3}\left(d^2 - 1 + 3\overline{d}(\overline{d}+1-d)\right). \tag{C.6}$$

Appendix D

Proof of Result 4

Let us take three arbitrary orthonormal bases $\{|\psi_1^1\rangle, |\psi_2^1\rangle\}$, $\{|\psi_1^2\rangle, |\psi_2^2\rangle\}$, and $\{|\psi_1^3\rangle, |\psi_2^3\rangle\}$ in $\mathbb{C}^2$ such that $M_{x_y|y} = |\psi_{x_y}^y\rangle\langle\psi_{x_y}^y|$ with $x_y \in [2]$ for all $y \in \{1,2,3\}$. A unitary can always be applied to these three measurements. Therefore, without any loss of generality, we can assume that

$$|\psi_{x_1}^1\rangle\langle\psi_{x_1}^1| = \frac{1}{2}\left[\mathbb{1} + (-1)^{x_1}\sigma_z\right] \text{ with } x_1 \in [2], \tag{D.1}$$

$$|\psi_{x_2}^2\rangle\langle\psi_{x_2}^2| = \frac{1}{2}\left[\mathbb{1} + (-1)^{x_2}\left(\alpha\sigma_z + \sqrt{1-\alpha^2}\sigma_x\right)\right] \text{ with } x_2 \in [2], \tag{D.2}$$

$$|\psi_{x_3}^3\rangle\langle\psi_{x_3}^3| = \frac{1}{2}\left[\mathbb{1} + (-1)^{x_3}\left(\beta\sigma_z + \gamma\sqrt{1-\beta^2}\sigma_x \pm \sqrt{1-\beta^2}\sqrt{1-\gamma^2}\sigma_y\right)\right] \text{ with } x_3 \in [2]. \tag{D.3}$$

where $-1 \leq \alpha, \beta, \gamma \leq 1$.

Due to the same reasoning as for (A.1), the maximum average success probability for the above-mentioned given set of three rank-one projective qubit measurements is given by,

$$S_c(n=3, \overline{d}=2, d=2) = \frac{1}{24}\sum_{x_1,x_2,x_3=1}^{2} ||M_{x_1|1} + M_{x_2|2} + M_{x_3|3}||. \tag{D.4}$$

By definition, $||M_{x_1|1} + M_{x_2|2} + M_{x_3|3}||$ is the maximum eigenvalue of $(M_{x_1|1} + M_{x_2|2} + M_{x_3|3})$, which can be evaluated easily. Subsequently, it can be checked that

$$\sum_{x_1,x_2,x_3=1}^{2} ||M_{x_1|1} + M_{x_2|2} + M_{x_3|3}|| = 12 + \sqrt{3 + 2\alpha - 2\beta - 2\alpha\beta - 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}$$
$$+ \sqrt{3 - 2\alpha + 2\beta - 2\alpha\beta - 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}$$
$$+ \sqrt{3 - 2\alpha - 2\beta + 2\alpha\beta + 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}$$
$$+ \sqrt{3 + 2\alpha + 2\beta + 2\alpha\beta + 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}. \tag{D.5}$$

We have found out the minimum of the above expression (D.5) by performing numerical optimization. It turns out that

$$\min_{\alpha,\beta,\gamma\in[-1,1]} \left(\sum_{x_1,x_2,x_3=1}^{2} ||M_{x_1|1} + M_{x_2|2} + M_{x_3|3}|| \right) = 18. \tag{D.6}$$

In other words,

$$\min_{\alpha,\beta,\gamma\in[-1,1]} (\xi_1 + \xi_2 + \xi_3 + \xi_4) = 6, \tag{D.7}$$

where

$$\xi_1 = \sqrt{3 + 2\alpha - 2\beta - 2\alpha\beta - 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}},$$
$$\xi_2 = \sqrt{3 - 2\alpha + 2\beta - 2\alpha\beta - 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}},$$
$$\xi_3 = \sqrt{3 - 2\alpha - 2\beta + 2\alpha\beta + 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}},$$
$$\xi_4 = \sqrt{3 + 2\alpha + 2\beta + 2\alpha\beta + 2\gamma\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}.$$

In order to prove Result 4, it is sufficient to show that $(\xi_1 + \xi_2 + \xi_3 + \xi_4) = 6$ only if the three projective measurements are compatible, or $(\alpha, \beta, \gamma) = \{(\pm 1/2, \pm 1/2, \pm 1)\}$. Since $-1 \leq \alpha, \beta, \gamma \leq 1$, we divide the regions of α, β and γ into the following sub-regions:

i. $\alpha, \beta, \gamma \in [0, 1]$,

ii. $\alpha \in [-1,0]$; $\beta,\gamma \in [0,1]$,

iii. $\alpha,\beta \in [-1,0]$; $\gamma \in [0,1]$,

iv. $\alpha,\beta,\gamma \in [-1,0]$,

v. $\alpha,\gamma \in [0,1]$; $\beta \in [-1,0]$,

vi. $\alpha,\gamma \in [-1,0]$; $\beta \in [0,1]$,

vii. $\alpha,\beta \in [0,1]$; $\gamma \in [-1,0]$,

viii. $\alpha \in [0,1]$; $\beta,\gamma \in [-1,0]$.

We start by considering the above-mentioned sub-region (i), i.e., $\alpha,\beta,\gamma \in [0,1]$. In this case, we note the following holds from numerical evaluation,

$$\min_{\alpha,\beta,\gamma\in[0,1]} (\xi_1 + \xi_2) = 2, \tag{D.8}$$

and

$$\min_{\alpha,\beta,\gamma\in[0,1]} (\xi_3) \geqslant \min_{\alpha,\beta\in[0,1]} \sqrt{3 - 2\alpha - 2\beta + 2\alpha\beta} = 1, \tag{D.9}$$

since the derivative of the above expression is zero at $\alpha = 1$ or/and $\beta = 1$.

Next, we evaluate the maximum as well as minimum of $(\xi_1 + \xi_2 + \xi_3)$ numerically under the constraint that $(\xi_1 + \xi_2 + \xi_3 + \xi_4) = 6$. It is obtained that

$$\min_{\alpha,\beta,\gamma\in[0,1]} (\xi_1 + \xi_2 + \xi_3) = \max_{\alpha,\beta,\gamma\in[0,1]} (\xi_1 + \xi_2 + \xi_3) = 3, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6. \tag{D.10}$$

Therefore, we have that

$$\xi_1 + \xi_2 + \xi_3 = 3, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6. \tag{D.11}$$

Hence, the following is implied from (D.8), (D.9), (D.11),

$$\xi_1 + \xi_2 = 2, \ \xi_3 = 1, \text{ and } \xi_4 = 3, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6. \tag{D.12}$$

Next, it can be checked that $\xi_3 = 1$ only if $\alpha = 1$ or/and $\beta = 1$. Now, when $\alpha = 1$, then $\xi_4 = 3$ implies that $\beta = 1$. Similarly, when $\beta = 1$, then $\xi_4 = 3$ implies that $\alpha = 1$. Therefore, when $\alpha, \beta, \gamma \in [0,1]$, then $(\xi_1 + \xi_2 + \xi_3 + \xi_4) = 6$ holds only if $\alpha = \beta = 1$.

Next, consider the sub-region (iii), i.e., for $\alpha, \beta \in [-1,0]$; $\gamma \in [0,1]$. We note that if $\alpha \to -\alpha$ and $\beta \to -\beta$ then the four expressions ξ_i interchange among themselves as we can readily verify $\xi_1 \to \xi_2$, $\xi_2 \to \xi_1$, $\xi_3 \to \xi_4$, $\xi_4 \to \xi_3$. Thus, following a similar calculation as for sub-region (i), we find that $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 6$ holds only if $\alpha = \beta = -1$. Similarly, for sub-regions (vi) and (viii), one can show that $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 6$ holds only if $-\alpha = \beta = 1$ and $\alpha = -\beta = 1$, respectively.

Next, let us focus on the sub-region (iv), i.e., when $\alpha, \beta, \gamma \in [-1,0]$. In this case, we obtain the following by performing numerical optimizations,

$$
\begin{aligned}
\min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_1 + \xi_4) &= \max_{\alpha,\beta,\gamma\in[-1,0]} (\xi_1 + \xi_4) = 2, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6, \\
\min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_2 + \xi_4) &= \max_{\alpha,\beta,\gamma\in[-1,0]} (\xi_2 + \xi_4) = 2, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6, \\
\min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_1 + \xi_3) &= \max_{\alpha,\beta,\gamma\in[-1,0]} (\xi_1 + \xi_3) = 4, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6, \\
\min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_2 + \xi_3) &= \max_{\alpha,\beta,\gamma\in[-1,0]} (\xi_2 + \xi_3) = 4, \text{ when } \xi_1 + \xi_2 + \xi_3 + \xi_4 = 6.
\end{aligned}
$$

Hence, we can infer that whenever $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 6$,

$$\xi_1 + \xi_4 = \xi_2 + \xi_4 = 2, \quad \xi_1 + \xi_3 = \xi_2 + \xi_3 = 4. \tag{D.13}$$

Therefore, we have $\xi_1 = \xi_2$ if $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 6$. Also, it can be easily checked from the expressions of ξ_1, ξ_2, ξ_3 and ξ_4 that

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 12. \tag{D.14}$$

By putting $\xi_1 = \xi_2 = \xi$, $\xi_3 = 4 - \xi$, $\xi_4 = 2 - \xi$, we get from (D.14) that

$$2\xi^2 + (4 - \xi)^2 + (2 - \xi)^2 = 12. \tag{D.15}$$

The possible solutions of the above equation are $\xi = 1$ and $\xi = 2$.
Before proceeding, let us point out the following observations that can be checked numerically,

$$\min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_1) = \min_{\alpha,\beta,\gamma\in[-1,0]} (\xi_2) = 1. \tag{D.16}$$

First, we take $\xi = 1$. It can be shown that $\alpha, \beta, \gamma \in [-1, 0]$, $\xi_1 = \xi_2 = 1$ only if $\alpha = -1$ and $\beta = -1$. Next, let us take $\xi = 2$. Consequently, we have that $\xi_1 = \xi_2 = 2$, $\xi_3 = 2$, $\xi_4 = 0$. It can be checked that the unique solution of these four equations is given by, $\alpha = -1/2$, $\beta = -1/2$, $\gamma = -1$.

Next, we remark that for the remaining sub-regions (ii), (v), and (vii) wherein the variables α, β, γ changes their signs with respect to the sub-region (iv) where $\alpha, \beta, \gamma \in [-1, 0]$, the four expressions ξ_i interchange among themselves. Thus, a similar calculation applies to these three regions, and, consequently, the solution for $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 6$ is the same with the appropriate signs.

Finally, let us note that there are, in general, two cases where we do not observe any advantage. Firstly, $\alpha = \pm 1$ and $\beta = \pm 1$, which implies that the three measurements $\{M_{x_1|1}\}$, $\{M_{x_2|2}\}$ and $\{M_{x_3|3}\}$ are compatible. Secondly, $(\alpha, \beta, \gamma) = \{(\pm 1/2, \pm 1/2, -1), (\pm 1/2, \mp 1/2, 1)\}$, which are obtained in sub-regions (iv), (ii), (v), and (vii), implies that the three measurements are incompatible. This completes the proof.

Appendix E

Calculation of the lower bound of the sharpness parameters

The condition for Alice^1 and Bob^1 to witness entanglement is given by,

$$\xi_1 \lambda_1 > \frac{1}{3} \tag{E.1}$$

Similarly, the condition for Alice^2 and Bob^2 to witness entanglement is given by,

$$\xi_2 \lambda_2 > \frac{3}{\left(1+2\sqrt{1-\xi_1^2}\right)\left(1+2\sqrt{1-\lambda_1^2}\right)}. \tag{E.2}$$

Now, in order to ensure maximum number of sequential observers witnessing entanglement, Alice^1 and Bob^1 should choose the sharpness parameters of their measurements in such a way that these can detect entanglement causing minimal disturbance to the state. In any unsharp measurement of the form (4.3) on qubits, the disturbance can be reduced by reducing the associated sharpness parameter [143, 148]. Hence, Alice^1 and Bob^1 should choose sharpness parameters satisfying the following relation,

$$\xi_1 \lambda_1 = \frac{1}{3} + \epsilon_1 \text{ with } 0 < \epsilon_1 << 1. \tag{E.3}$$

Next, the sharpness parameters of the measurements by Alice^2 and Bob^2 also should ensure detection of entanglement causing minimal disturbance to the state. For this, we have to minimize the right hand side of (E.2) under the constraint given by Eq.(E.3). In other words, we have to perform the following optimization problem,

$$\begin{aligned}
&\max_{\xi_1,\lambda_1} f_1(\xi_1,\lambda_1)\\
&\text{such that} \quad \xi_1\lambda_1 = \frac{1}{3}+\epsilon_1 \quad \text{and} \quad 0<\xi_1,\lambda_1\leq 1\\
&\text{where } f_1(\xi_1,\lambda_1) = \left(1+2\sqrt{1-\xi_1^2}\right)\left(1+2\sqrt{1-\lambda_1^2}\right).
\end{aligned} \tag{E.4}$$

Now, taking $\epsilon_1 = 10^{-2}$, we obtain $\xi_1 = \lambda_1 = 0.58$. Hence, the condition (E.2) becomes

$$\xi_2\lambda_2 > 0.43. \tag{E.5}$$

(Note that all numerical values appearing in this appendix are rounded to two decimal places.)

Next, we find out the conditions under which Alice^1-Bob^1, Alice^2-Bob^2 and Alice^3-Bob^3 can witness entanglement in such a way that the measurement by each observer causes the minimum possible disturbance to the state. Alice^3-Bob^3 can detect entanglement if

$$\xi_3\lambda_3 > \frac{27}{\prod_{i=1}^{2}\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right)}. \tag{E.6}$$

To ensure minimum possible disturbance by Alice^3-Bob^3 while witnessing entanglement, we minimize the right-hand side of (E.6), performing the following optimization problem,

$$\begin{aligned}
&\max_{\xi_2,\lambda_2} f_2(\xi_2,\lambda_2)\\
&\text{such that} \quad \xi_2\lambda_2 = 0.43+\epsilon_2, \text{ with } 0<\epsilon_2<<1, \quad 0<\xi_2,\lambda_2\leq 1\\
&\text{where } f_2(\xi_2,\lambda_2) = \prod_{i=1}^{2}\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right) \quad \text{with} \quad \xi_1=\lambda_1=0.58.
\end{aligned} \tag{E.7}$$

Taking $\epsilon_2 = 10^{-2}$, we obtain $\xi_2 = \lambda_2 = 0.66$. With these values, the condition (E.6) becomes

$$\xi_3\lambda_3 > 0.62. \tag{E.8}$$

Proceeding similarly, we check the conditions on the parameters under which Alice^1-Bob^1, Alice^2-Bob^2, Alice^3-Bob^3 and Alice^4-Bob^4 can witness entanglement, given by

$$\xi_4 \lambda_4 > \frac{243}{\prod_{i=1}^{3}\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right)}. \tag{E.9}$$

Here the corresponding optimization problem is,

$$\begin{aligned}
&\max_{\xi_3,\lambda_3} f_3(\xi_3,\lambda_3) \\
&\text{such that} \quad \xi_3 \lambda_3 = 0.62+\epsilon_3, \text{ with } 0<\epsilon_3<<1, \quad 0<\xi_3,\lambda_3 \leq 1 \\
&\text{where } f_3(\xi_3,\lambda_3) = \prod_{i=1}^{3}\left(1+2\sqrt{1-\xi_i^2}\right)\left(1+2\sqrt{1-\lambda_i^2}\right) \text{ with} \\
&\qquad\qquad \xi_1=\lambda_1=0.58,\ \xi_2=\lambda_2=0.66.
\end{aligned} \tag{E.10}$$

Taking $\epsilon_3 = 10^{-2}$, we get $\xi_3 = \lambda_3 = 0.79$. However, with these values, Eq.(E.9) becomes

$$\xi_4 \lambda_4 > 1.13. \tag{E.11}$$

Since $\xi_4, \lambda_4 \in (0,1]$, the above condition cannot be satisfied. Therefore, at most three pairs (Alice^1-Bob^1, Alice^2-Bob^2, Alice^3-Bob^3) can detect entanglement.

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